國立高雄大學九十三學年度研究所碩士班招生考試試題

系所組別：統計學研究所　　　　　　　　　　　　　　科目：機率論

1. Assume X has a normal distribution with mean zero and variance σ^2. Show that $E(X^4) = 3(E(X^2))^2$. (10%)

2. Independent trials, each of which is a success with probability p, are performed until there are k consecutive successes. Let N_k denote the number of necessary trials. Find the expectation of N_k, i.e. $E(N_k)$. (10%)

3. Consider an autoregressive model,

$$X_n = -X_{n-1} + \varepsilon_n, \text{ for } n = 1, 2, \cdots,$$

where $\varepsilon_i, i = 1, 2, \cdots$, are i.i.d. random variables with $E(\varepsilon_i) = \mu$ and $Var(\varepsilon_i) = \sigma^2$, and $X_0 = 0$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Show that $\sqrt{n}(\bar{X}_n - \mu/2)$ converges in distribution to a normal distribution with mean zero and variance $\sigma^2/2$ as $n \to \infty$. (10%)

4. Let X be a Poisson distribution with a parameter λ. Show that $(X - \lambda)/\sqrt{\lambda}$ converges in distribution to a normal distribution with mean zero and variance one as $\lambda \to \infty$. (10%)

5. Let $X_1, X_2, \cdots$, be a sequence of random variables such that X_1 is uniform$[0, 1]$ and where for $n = 1, 2, \cdots$, the conditional distribution of X_{n+1} given $X_1, \cdots, X_n$ is uniform$[0, cX_n]$ for some number c such that $\sqrt{3} < c < 2$. Find the expectation of X_n^r for $r > 0$. Then show that $E(X_n) \to 0$ as $n \to \infty$ but $E(X_n^2) \not\to 0$ as $n \to \infty$. (20%)

6. **(a)** Let X be a random variable, and $g(x)$ is a convex function. Prove the following inequality (Jensen's Inequality)

$$E(g(X)) \geq g(E(X)). \quad (10\%)$$

   (b) If $y_1, \cdots, y_n$ are positive numbers, define

$$y_A = \frac{1}{n}(y_1 + \cdots + y_n), y_G = [y_1 y_2 \cdots y_n]^{1/n}, y_H = \frac{1}{\frac{1}{n}(\frac{1}{y_1} + \cdots + \frac{1}{y_n})}.$$

   Use Jensen's inequality to prove that $y_H \leq y_G \leq y_A$. (15%)

7. Assume $X_1, \cdots, X_n$ are i.i.d. uniform$(0, a)$, $a > 0$. Let $X_{(1)}, \cdots, X_{(n)}$ denote the order statistics of $X_1, \cdots, X_n$. Define $R = X_{(n)} - X_{(1)}$ and $V = (X_{(1)} + X_{(n)})/2$. Find the joint distribution of (R, V) and the marginal distributions of R and V. (15%)