

國立高雄大學九十三年度研究所碩士班招生考試試題

系(所)別:統計研究所

科目:基礎數學

滿分為100分。作答時請於答案卷上註明題號及附上推導過程。

1. Suppose $h(t)$ is a nondecreasing continuous function for $t \geq 0$. Let $H(t) = \int_0^t h(x)dx$. Is $\frac{H(t)}{t}$ nondecreasing for $t > 0$? Justify your answer. (10%)

2. Which is larger (for $n > 8$): $(\sqrt{n})^{\sqrt{n+1}}$ or $(\sqrt{n+1})^{\sqrt{n}}$? (12%)

3. (a) Evaluate $\lim_{x \rightarrow 3} \frac{x}{x-3} \int_3^x \frac{\sin t}{t} dt$. (6%)

(b) Evaluate $\lim_{n \rightarrow \infty} \int_0^1 \frac{ny^{n-1}}{1+y} dy$. (10%)

4. Define a sequence by

$$a_n = \int_0^1 (1-x^2)^n dx.$$

Show that $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$. Does $\sum_{n=1}^{\infty} a_n$ converge? (12%)

5. Let $M_{n \times n}(\mathbf{R})$ be the set of all $n \times n$ matrices over \mathbf{R} .

(a) Show that $\{I, A, A^2, \dots, A^n\}$ is linearly dependent for all $A \in M_{n \times n}(\mathbf{R})$. (5 %)

(b) Let $A \in M_{n \times n}(\mathbf{R})$. Show that A is invertible if and only if I belongs to the span of $\{A, A^2, \dots, A^n\}$. (7 %)

6. Let A be an $n \times n$ real skew-symmetric matrix; i.e., $A^t = -A$, where A^t denotes the transpose of A .

(a) Show that $\det(A) - (-1)^n \det(A) = 0$. (4 %)

(b) If n is odd, show that A must be singular. (3 %)

(c) What can you say about the eigenvalues of A ? (6 %)

7. Let $T : M_{n \times n}(\mathbf{F}) \rightarrow M_{n \times n}(\mathbf{F})$ be the linear transformation defined by $T(A) = \frac{A + A^t}{2}$, where $M_{n \times n}(\mathbf{F})$ denotes the set of all $n \times n$ matrices over a field \mathbf{F} .

(a) What is $\text{Ker}(T)$ (kernel of T)? (6 %)

(b) What is the dimension of $\text{Ker}(T)$? (7 %)

8. Find all possible 2×2 real matrices A such that

$$A^4 = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}. \quad (12\%)$$