國立高雄大學九十四學年度研究所碩士班招生考試試題

系所組別：統計學研究所　　　　　　　　　　科目：基礎數學

考試時間：100 分鐘　　　　　　　　　　本科原始成績滿分 100 分

1. (10 points) Show that $(1+x)^{1/x} = e\cdot(1-\frac{x}{2}+\frac{11x^2}{24}+o(x^2))$ as $x\to 0$.

2. (10 points) Prove that $\lim_{x\to 0}(1+ax)^{1/x} = e^a$ for every real a.

3. (16 points) Let $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = \int_n^{n+1}\frac{dx}{x}$ for $n = 1, 2, 3, \cdots$.

   (a) Show the series $s_n = \sum_{i=1}^{n}(-1)^{i-1}a_i$ converges. In fact, s_n converges to *Euler's constant.*

   (b) Use the result of (a) to show $\sum_{k=1}^{\infty}(-1)^{k-1}/k = \log 2$.

4. (12 points) Evaluate

   (a) $\int_0^{\pi/2}\sin^2 x dx$

   (b) $\int_0^{\pi/2}\sin^4 x dx$

5. (12 points) Consider the $p\times p$ matrix $X'X$, and X' is the transpose of X. Let x' be the ith row of X. Hence $X'X - xx'$ is the $X'X$ matrix with the ith row removed. Assume $(X'X)^{-1}$ exists. Show that

$$(X'X - xx')^{-1} = (X'X)^{-1} + \frac{(X'X)^{-1}xx'(X'X)^{-1}}{1-x'(X'X)^{-1}x}.$$

6. (10 points) Define the 2×2 matrix, $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$. Find a 2×2 matrix B such that $A = BB'$.

7. (20 points) Let the 4×4 matrix C be defined by

$$C = \begin{pmatrix} 2 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{pmatrix}.$$

   Find the determinant of C and the inverse matrix of C.

8. (10 points) Let A be a $p\times p$ idempotent matrix and let B be a $p\times p$ tripotent matrix. That is

$$A = A^2 \text{ and } B = B^3.$$

   Find the all possible eigenvalues of A and B.