國立高雄大學九十四學年度研究所碩士班招生考試試題

系所組別：統計學研究所　　　　　　　　　　　　科目：數理統計

考試時間：100分鐘　　　　　　　　　　　　本科原始成績滿分 100分

試卷共有兩張。作答時請於答案卷上註明題號及附上推導過程。

1. Let X_1, X_2 and X_3 be independent random variables with mean δ, 2δ and 3δ, respectively, and the same variance σ^2. The following three estimates are all unbiased for δ:

$$\hat{\delta}_1 = \frac{1}{3}\left(X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3\right), \ \hat{\delta}_2 = \frac{1}{6}\left(X_1 + X_2 + X_3\right), \ \hat{\delta}_3 = \frac{1}{14}\left(X_1 + 2X_2 + 3X_3\right).$$

(a) Which of these three estimators has the smallest variance? (6 %)

(b) Is there a linear unbiased estimator of δ with smaller variance than any of the three suggested above? Justify your answer. (10 %)

2. Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & \theta \le x < \infty, \\ 0\ , & \text{otherwise}, \end{cases}$$

where $-\infty < \theta < \infty$.

(a) Find the MLE (maximum-likelihood estimator) of θ. (8 %)

(b) Find an estimator of θ by the method of moments and show that it is consistent. (12 %)

3. Let $X_1, \ldots, X_n$ be a random sample from the Poisson density

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!},\ x = 0, 1, \ldots,$$

where $\theta > 0$.

(a) Show that $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic. (5 %)

(b) Let $Y = (-1)^{X_1}$. Show that Y is unbiased for $e^{-2\theta}$ and conclude that it is an unreasonable estimator. (5 %)

(c) Based on the results of (a) and (b), find an UMVUE (uniformly minimum-variance unbiased estimator) for $e^{-2\theta}$. (12 %)

4. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from the density

$$f(x;\theta) = \begin{cases} \dfrac{3x^2}{\theta^3}, & 0 < x < \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(k\theta < Y_4 < \theta)$, where $0 < k < 1$. (10 %)

(b) Find a 95% confidence interval for θ. (6 %)

5. Let X be a single observation from the density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

(a) In testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the power function and size of the test given by the following: Reject H_0 if and only if $X \geq \frac{1}{2}$. (8 %)

(b) Find a most powerful size-α test of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$. (8 %)

(c) Among all possible (simple) likelihood-ratio tests of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$, find that test that minimizes $\alpha + \beta$, where α and β are the respective sizes of the Type I and Type II errors. (10 %)