國立高雄大學九十五學年度博士班招生考試試題

科目：機率論　　　　　　系所：統計學研究所　　　　是否使用計算機：是
考試時間：100 分鐘　　　本科原始成績：100 分

1. Let $X, X_1, X_2, \cdots$ be random vectors with values in $\mathbf{R}^d$.

   (i) Write down the definitions of (1) X_n converges in law to X, $X_n \xrightarrow{d} X$, (2) X_n converges in probability to X, $X_n \xrightarrow{p} X$, (3) X_n converges in the rth mean to X, $X_n \xrightarrow{r} X$, and (4) X_n converges almost surely to X, $X_n \xrightarrow{a.s.} X$. (6%)

   (ii) Give an example to show that $X_n \xrightarrow{d} X$ does not imply $X_n \xrightarrow{p} X$. (6%)

   (iii) Give an example to show that $X_n \xrightarrow{r} X$ for all $r > 0$ does not imply that $X_n \xrightarrow{a.s.} X$. (6%)

   (iv) Give an example to show that $X_n \xrightarrow{a.s.} X$ does not imply $X_n \xrightarrow{r} X$ for any $r > 0$. (6%)

2. Suppose that the random variable X has mean m and variance σ^2. Prove

$$P(X - m \geq \alpha) \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}, \quad \alpha \geq 0. (12\%)$$

3. Let $U_1, U_2, \cdots$ be a sequence of independent uniform $(0, 1)$ random variables, and let

$$N = \min\{n \geq 2 : U_n > U_{n-1}\} \text{ and } M = \min\{n \geq 1 : U_1 + \cdots + U_n > 1\}.$$

   Show that N and M have the same probability distribution and find their common mean. (12%)

4. Show that if $\{X_n\}$ and $\{Y_n\}$ are two independent sequences of random vectors, and if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} X \\ Y \end{pmatrix},$$

   where X and Y are taken to be independent. (10%)

5. Let $X_1, X_2, \cdots$ be a sequence of random variables such that X_1 is uniform on $[0, 1]$, and where for $n = 1, 2, \cdots$, the conditional distribution of X_{n+1} given $X_1, \cdots, X_n$ in uniform on $[0, cX_n]$ for a fixed number c such that $\sqrt{3} < c < 2$.

   (i) Show that X_n converges to 0 in mean ($r = 1$) but not in quadratic mean ($r = 2$). (12%)

   (ii) Does X_n converge to 0 almost surely? (12%)

6. Suppose $X_1, X_2, \cdots, X_n$ is a sample from the uniform distribution on $(0, \theta)$. Let M_n be the maximum of the sample. Let $Z_n = n(\theta - M_n)$.

   (i) Show that $Z_n \xrightarrow{d} Z$, where Z has the exponential distribution. (12%)

   (ii) Find the asymptotic distribution of $\frac{n+c}{n} M_n$ for some positive number c. (6%)