國立高雄大學九十五學年度研究所碩士班招生考試試題

科目：數理統計

考試時間：　100　分鐘　　　　　　　　本科原始成績：滿分　100 分

1. Let f_0 and f_1 be two probability density functions. The Kullback-Leibler information number is defined as

$$K(f_0, f_1) = E_0 \log \frac{f_0(X)}{f_1(X)} = \int \log \frac{f_0(x)}{f_1(x)} f_0(x) dx.$$

Show that $K(f_0, f_1) \geq 0$. (12 points)

2. Suppose $X_1, \cdots, X_n$ are i.i.d. uniform observations on the interval $(\theta, \theta+1)$, $-\infty < \theta < \infty$. Find a minimal sufficient statistic for θ. (10 points)

3. Suppose $2n$ random variables, $X_1, Y_1, X_2, Y_2, \cdots, X_n, Y_n$, are independent, and for each i, $i = 1, \cdots, n$, X_i and Y_i follow the normal distribution with mean μ_i and variance σ^2, i.e. $N(\mu_i, \sigma^2)$. Here the parameters, $\mu_1, \cdots, \mu_n$ and σ^2, are assumed to be unknown.

   (a) Find the MLE of σ^2, $\hat{\sigma}^2$, and show that $\hat{\sigma}^2$ is a biased estimator of σ^2. (12 points)

   (b) Based on $\hat{\sigma}^2$, find the best unbiased estimator (UMVUE) of σ^2. (12 points)

4. Let $X_1, \cdots, X_n$ be a sample from a $N(\mu_1, \sigma^2)$ and $Y_1, \cdots, Y_m$ be another independent sample from a $N(\mu_2, \rho^2\sigma^2)$, where ρ is known. Define $\bar{X} = \sum_{i=1}^{n} X_i/n$; $\bar{Y} = \sum_{i=1}^{m} Y_i/m$; $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and $S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$. Show that

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{n} + \frac{\rho^2}{m}}\sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2/\rho^2}{n+m-2}}}$$

has a t distribution with $n + m - 2$ degrees of freedom and $S_Y^2/(\rho^2 S_X^2)$ has an F distribution with $m-1$ and $n-1$ degrees of freedom. (20 points)

5. Suppose that $X_1, \cdots, X_n$ are i.i.d. with a beta $(\mu,\ 1)$ pdf and $Y_1, \cdots, Y_m$ are i.i.d. with a beta$(\theta,1)$ pdf. Also assume that the Xs are independent of the Ys.

   (a) Find an likelihood ratio test (LRT) of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$, and show that this LRT can be based on

$$T = \frac{\sum \log X_i}{\sum \log X_i + \sum \log Y_j}.$$

   (12 points)

   (b) When H_0 is true, find the distribution of T, and then show how to get a test of size $\alpha = 0.10$. (10 points)

6. Let $X_1, \cdots, X_n$ be a sample with pdf $f(x|\theta) = \theta \exp(-\theta x), x > 0$. Here the parameter θ is unknown and $\theta > 0$. Now we are interesting in testing

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta < \theta_0.$$

Find the UMP (uniformly most powerful) level α test. (12 points)