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Mathematical Statistics Qualifying Test

1. Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$. Show that the probability density function of $(X_1 - \overline{X})/S$ is

$$f(z) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)} \sqrt{\frac{n}{n-1}} \left(1 - \frac{nz^2}{n-1}\right)^{(n/2)-2}, \qquad \text{if } 0 < |z| < \sqrt{\frac{n-1}{n}},$$

where $\overline{X}$ is the sample mean, $S^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2$. (20%)

2. Let $X_1, \cdots, X_n$ be iid $N(\theta, \theta^2)$, $\theta > 0$.

   (a) Prove that for any number a the estimator $a\overline{X} + (1-a)(cS)$ is an unbiased estimator of θ, where $c = \frac{\sqrt{n-1}\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}$, $\overline{X}$ is the sample mean, S^2 is the sample variance.

   (b) Find the value of a that produces the estimator with minimum variance.

   (c) Show that $(\overline{X}, S^2)$ is a sufficient statistic for θ but it is not a complete sufficient statistic. (20%)

3. Let $X_1, \cdots, X_n$ be a sample from a population with the density

$$f(x, \theta) = \theta(\theta+1)x^{\theta-1}(1-x), \quad 0 < x < 1, \quad \theta > 0.$$

   (a) Show that $T_n = \frac{2\overline{X}}{1-\overline{X}}$ is a method of moments estimate of θ.

   (b) Show that

$$\frac{\sqrt{n}(T_n - \mu_n(\theta))}{\sigma_n(\theta)} \to N(0,1)$$

   in law where $\mu_n(\theta) = \theta$, $\sigma_n^2 = \theta(\theta+2)^2 / 2(\theta+3)$.

   (c) Show that T_n is not efficient by calculating the Cramér-Rao lower bound. (20%)

4. Let X_1, X_2 be iid uniform $(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$
$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

(a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

(b) Calculate the power function of each test.

(c) Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

(d) Show how to get a test that has the same size but is more powerful than ϕ_2. (20%)

5. Let $X_1, \cdots, X_n$ be a sample from a population with mean μ and variance $\sigma^2 < \infty$. Suppose h has a second derivative h'' continuous at μ and that $h'(\mu) = 0$.

(a) Show that $\sqrt{n}[h(\overline{X}) - h(\mu)] \xrightarrow{P} 0$ while $n[h(\overline{X}) - h(\mu)]$ is

asymptotically distributed as $\frac{1}{2}h''(\mu)\sigma^2 V$ where $V \sim \chi_1^2$.

(b) Use the (a) part to show that when $\mu = \frac{1}{2}$, $n[\overline{X}(1-\overline{X}) - \mu(1-\mu)] \xrightarrow{L} -\sigma^2 V$ with $V \sim \chi_1^2$. Give an approximation to the distribution of $\overline{X}(1-\overline{X})$ in terms of the χ_1^2 distribution function when $\mu = \frac{1}{2}$. (20%)