Optimal Minimax Designs for Three Different Criteria

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三種大中取小準則下之最適設計

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摘要

在本論文中，我們討論異方差模型中大中取小的最適設計。此論文的目的為比較三個大中取小準則。首先，根據Wong(1998)的演算法，我們先求得針對這三個不同準則下的最適設計，並且比較它們的相對效率。最後，對於簡單線性模型，在一些簡單的條件下，我們證明這三個大中取小的最適設計是相等的。

關鍵字 : 近似設計，黃金比例方法，異方差模型，Powell’s 方法
Optimal Minimax Designs for Three Different Criteria Designs

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ABSTRACT

In this thesis, we are interested in finding optimal minimax designs for heteroscedastic model. The goal of this thesis is to study three minimax criteria. Here for different efficiency functions, the corresponding numerically optimal minimax designs are computed according to the generating algorithm of Wong (1998), and then their relative efficiencies are compared. Finally for simple linear model, under some simple conditions, we show these three types of optimal minimax designs are equivalent.

Keywords: Approximate design; Golden Section method; Heteroscedastic model; Powell’s method
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1 Introduction

In this thesis, the polynomial regression model on a compact design space, \( \mathcal{X} \), is represented as

\[
y(x) = f^T(x)\beta + e(x)/\sqrt{\lambda(x)},
\]

where \( f^T(x) = (1, x, \ldots, x^d) \) is linearly independent continuous regression functions defined on \( \mathcal{X} \) in \( \mathbb{R} \), \( \beta^T = (\beta_0, \beta_1, \ldots, \beta_d) \) is the vector of unknown parameters, the efficiency function \( \lambda(x) \) is a known, bounded, positive real-valued continuous function defined on \( \mathcal{X} \), and \( e(x) \) is a real-valued error variables having mean 0 and constant variance, \( \sigma^2 \).

Uncorrelated observations on the dependent variable \( y \) taken \( x_1, \ldots, x_d \in \mathcal{X} \), where the \( x_i \)'s are not necessarily distinct, are represented by real random variables \( y(x_1), \ldots, y(x_d) \).

Usually, a design \( \pi \) is treated as a probability measure on \( \mathcal{X} \) with positive mass \( w_i \) at the distinct support point \( x_i \in \mathcal{X}, i = 1, \ldots, n \), and \( \sum w_i = 1 \). Hence \( \pi \) is represented as

\[
\pi = \begin{cases} 
  x_1 & x_2 & \ldots & x_n \\
  w_1 & w_2 & \ldots & w_n 
\end{cases}.
\]

Let \( \hat{\beta} \) be the least square estimator of the unknown parameter vector \( \beta \). Then the inverse of covariance matrix of \( \hat{\beta} \) is proportional to the \((d + 1) \times (d + 1)\) information matrix of \( \pi \),

\[
M(\pi) = \int_{\mathcal{X}} \lambda(x)f(x)f^T(x)\pi(dx),
\]

and the variance function of \( \pi \) at a point \( x \) is defined as

\[
d(x, \pi) = f^T(x)M^{-1}(\pi)f(x).
\]

Usually an optimal design problem is characterized by the triplet \((\mathcal{X}, f(x), \lambda(x))\) together with a convex optimality criterion function \( \Phi \), i.e. a design \( \pi \) minimizes \( \Phi(M(\pi)) \). Here \( \Phi \) is chosen to have the following form

\[
\Phi(M(\pi)) = \max_{z \in \mathcal{Z}} \Psi(z, M(\pi))
\]

for some functional \( \Phi(\cdot) \) and \( \mathcal{Z} \) an arbitrary compact interval. That is, we are interested in finding optimal minimax designs.

When design space \( \mathcal{X} \) is \([-1, 1]\), the minimax design problems have been discussed in Gaylor and Sweeny (1965); Hoel and Levine (1964); Keifer and Wolfowitz (1964a), (1964b), (1965); Levine (1966); Wong (1992), (1994), and Brown and Wong (1996). In Gaylor and Sweeny (1965), they found optimal minimax designs for the simple linear model when \( \mathcal{Z} \) is a compact interval. Hoel and Levine (1964), Kiefer and Wolfowitz
(1964a), (1964b), (1965) obtained theoretical results when $Z = [-1,1]$, $[-1,t]$ or $[1,t]$ and $t$ is large. In all these studies, the models are homoscedastic, i.e. $\lambda(x) = 1$. When $\lambda(x)$ is not constant across $Z$, the corresponding minimax designs have been discussed in Kiefer and Wolfowitz (1964a), (1965), Hoel and Levine (1964) and Wong (1992), (1994), (1998).

Here we are interested in optimal minimax design problems for heteroscedastic model, i.e. $\lambda(x) \neq c$. Three different minimax criteria are considered. The first minimax criterion is G-optimal criterion, and a design $\pi^*$ is called G-optimal design if

$$\Phi_1(M(\pi)) = \max_{z \in Z} d(z, \pi),$$

and the minimization is taken over the set $\Pi$, containing all designs on $X$. That is, this design problem facing the statistician is to select a given number of uncorrelated observations from $X$ to estimate on $Z$ so that $\max_{z \in Z} d(z, \pi)$ has minimal variance among all designs on $X$. Wong (1992), (1994) found G-optimal designs when $Z = X$, and Wong (1998) discussed the cases when $Z \neq X$. In addition to this G-optimal criterion, the other two criteria are

$$\Phi_2(M(\pi)) = \max_{z \in Z} \{d(z, \pi) + 1/\lambda(z)\} \quad \text{and} \quad \Phi_3(M(\pi)) = \max_{z \in Z} \lambda(z)d(z, \pi).$$

For $\Phi_2$, since the predictive variance at $z$ is proportional to $d(z, \pi) + 1/\lambda(z)$, this design problem is to find an optimal design on $X$ with minimal predictive variance among all designs on $X$. Wong (1998) obtained the optimal designs of $\Phi_2$ for polynomial models and any compact interval $Z$. For $\Phi_3$, it can be considered as a generalized G-optimal criterion (Atkinson and Donev, 1992), and when $Z = X$, this criterion is equivalence to D-optimal criterion (a design minimizing the value of $| M(\pi) |^{-1}$). Here a design is called an MV-optimal design, if the design minimizes $\Phi_1(M(\pi))$ over $\Pi$; a design is called an MP-optimal design, if the design minimizes $\Phi_2(M(\pi))$ over $\Pi$; and a design is called an MD-optimal design, if the design minimizes $\Phi_3(M(\pi))$ over $\Pi$. We will denote an MV-optimal design by $\pi_{MV}$, an MP-optimal design by $\pi_{MP}$ and an MD-optimal design by $\pi_{MD}$. Since three different optimal minimax criteria are considered, the purpose of this thesis is to compare the optimal designs for these three criteria. To compare designs, we use optimal relative efficiencies. The $\Phi_i$-relative efficiency of design $\pi_A$ relative to $\pi_B$ is defined as

$$\Phi_i(M(\pi_A))/\Phi_i(M(\pi_B)),$$

$i = 1, 2, 3$. If this value is close to 1, these two designs are about equally informative in terms of the $\Phi_i$-criterion. If $\Phi_i(M(\pi_A))$ is larger than $\Phi_i(M(\pi_B))$, then $\pi_B$ furnishes more information than $\pi_A$ in terms of the $\Phi_i$-criterion.
This thesis is organized as follows. In Section 2, we discuss the methodology for constructing these three types of optimal minimax designs, and an algorithm proposed in Wong (1998) is introduced for generating the optimal minimax designs numerically. In Section 3, for simple linear regression model, the numerically optimal designs for three minimax criteria are given, and these designs are compared according to their relative efficiencies. From our numerical results, we show that for the simple linear model and some Z’s, \( \pi_{MV} \) and \( \pi_{MD} \) are equivalent, and MV- (MD-) optimal designs also minimize \( \Phi_2 \) under some conditions. In Section 4, we focus on quadratic and cubic polynomial models with symmetric convex efficiency functions. A conclusion is given in Section 5.

2 Equivalence theorem and generating algorithm for optimal minimax designs

Here we present a method for finding MV-, MP- and MD-optimal designs. First we would introduce the equivalence theorem for the optimal minimax design, and then based on this equivalence theorem, the generating algorithm in Wong (1998) is described.

2.1 Equivalence theorem

The key point for finding these optimal minimax designs is the equivalence theorem in Wong (1998). Let \( N \) be the set of all \( d \times d \) non-negative definite matrices. The equivalence theorem in Wong (1998) is given as follows.

\textbf{Theorem 2.1.} (Wong, 1998) Suppose we have a real-valued criterion \( H \) defined on \( N \times Z \) such that for each \( z \in Z \), the following assumptions are satisfied:

(a) \( H(A + B, z) \leq H(A, z) \) for all \( A, B \in N \),

(b) \( H(nM(\pi), z) = \phi(n)\Psi(M(\pi)) \) where the function \( \phi(n) \) is decreasing,

(c) the set \( \{\lambda(x)f(x)f^T(x); x \in \mathcal{X}\} \) is compact,

(d) \( \Psi(M((1-a)\pi_1 + a\pi_2), z) \leq (1-a)\Psi(M(\pi_1), z) + a\Psi(M(\pi_2), z) \) if \( 0 \leq a \leq 1 \) and all designs \( \pi_1 \) and \( \pi_2 \),

(e) \( \{\pi : \Psi(M(\pi), z) < \infty\} \neq \emptyset \),

(f) \( \Psi(M(\pi), z) \) is differentiable on any set: \( M_c = \{M(\pi) : \Psi(M(\pi), z) \leq c < \infty\} \).
Then a necessary and sufficient condition for a design \( \pi^* \) to minimize

\[
\max_{z \in \mathbb{Z}} \Psi(M(\pi), z)
\]

over \( \Pi \) is the existence of a probability measure \( \mu^* \) defined on \( A(\pi^*) \) such that

\[
c(x, \pi^*, \mu^*) \geq tr M(\pi^*) \left( \frac{\partial \Psi(M, u)}{\partial M} \right)_{M=M(\pi^*)} \mu^*(du) \quad \text{for all } x \in \mathcal{X}, \tag{1}
\]

with equality at the support points of \( \pi^* \). Here

\[
A(\pi) = \{ u \in \mathbb{Z} | \Psi(M(\pi), u) = \max_{z \in \mathbb{Z}} \Psi(M(\pi), z) \}
\]

and

\[
c(x, \pi, \mu) = tr \lambda(x) f(x) f^T(x) \left( \frac{\partial \Psi(M, u)}{\partial M} \right)_{M=M(\pi)} \mu(du).
\]

Thus, to apply this theorem to our three minimax design problems, we set

**MV-optimality:** \( H(M(\pi), z) = H_1(M(\pi), z) = \Psi_1(M(\pi), z) = d(z, \pi) \),

**MP-optimality:** \( H(M(\pi), z) = H_2(M(\pi), z) = \Psi_2(M(\pi), z) = d(z, \pi) + 1/\lambda(z) \),

**MD-optimality:** \( H(M(\pi), z) = H_3(M(\pi), z) = \Psi_3(M(\pi), z) = \lambda(z) d(z, \pi) \).

Wong (1998) had verified that the assumptions in Theorem 2.1 hold for \( H_i(M(\pi), z), \) \( i = 1, 2 \). For \( H_3 \), it is straightforward to verify that the assumptions in Theorem 2.1 hold, because \( H_3(M(\pi), z) = \lambda(z) H_1(M(\pi), z) \). Since the derivative of \( d(z, \pi) \) with respect to \( M \) is given by

\[
\frac{\partial d(z, \pi)}{\partial M} = -M(\pi)^{-1} f(z) f^T(z) M(\pi)^{-1},
\]

we can show that (1) reduces to

a. \( \pi^* \) is MV-optimal if and only if there exists a \( \mu^* \) on \( A_1(\pi^*) \) such that for all \( x \) in \( \mathcal{X} \),

\[
c_1(x, \mu^*, \pi^*) = \int_{A_1(\pi^*)} \lambda(x) g(x, u_1, \pi^*) \mu^*(du_1) - d(u_1, \pi^*) \leq 0. \tag{2}
\]

b. \( \pi^* \) is MP-optimal if and only if there exists a \( \mu^* \) on \( A_2(\pi^*) \) such that for all \( x \) in \( \mathcal{X} \),

\[
c_2(x, \mu^*, \pi^*)
\]

\[
= \int_{A_2(\pi^*)} \{1/\lambda(u_2) + \lambda(x) g(x, u_2, \pi^*)\} \mu^*(du_2) - d(u_2, \pi^*) - 1/\lambda(u_2) \leq 0. \tag{3}
\]
c. $\pi^*$ is MD-optimal if and only if there exists a $\mu^*$ on $A_3(\pi^*)$ such that for all $x$ in $\mathcal{X}$,

$$c_3(x, \mu^*, \pi^*) = \int_{A_3(\pi^*)} \lambda(u_3) \lambda(x) g(x, u_3, \pi^*) \mu^*(du_3) - \lambda(u_3)d(u_3, \pi^*) \leq 0. \tag{4}$$

Here $g(x, u, \pi) = (f^T(x)M(\pi)^{-1}f(u))^2$, $A_1(\pi) = \{u_1 \in Z| d(u_1, \pi) = \max_{z \in Z} d(z, \pi)\}$, $A_2(\pi) = \{u_2 \in Z| 1/\lambda(u_2) + d(u_2, \pi) = \max_{z \in Z} [1/\lambda(z) + d(z, \pi)]\}$ and $A_3(\pi) = \{u_3 \in Z| \lambda(u_3)d(u_3, \pi) = \max_{z \in Z} \lambda(z)d(z, \pi)\}$. Note that (i) $c_i(x, \mu^*, \pi^*)$ does not depend on $u_i$, $i = 1, 2, 3$. (ii) By definitions of $A_i(\pi)$, $i = 1, 2, 3$, since $u_i$ is an element in $A_i(\pi)$ and consequently, $d(u_i, \pi) = d(u_j, \pi)$ for any $u_i$ and $u_j$ in $A_1(\pi)$, $d(u_i, \pi) + 1/\lambda(u_i) = d(u_j, \pi) + 1/\lambda(u_j)$ for any $u_i$ and $u_j$ in $A_2(\pi)$ and $\lambda(u_i)d(u_i, \pi) = \lambda(u_j)d(u_j, \pi)$ for any $u_i$ and $u_j$ in $A_3(\pi)$.

### 2.2 Generating algorithm

To apply equivalence theorem to a wide variety of practical problems, it is essential to have a computer algorithm to search for these optimal minimax designs numerically, because it is hard to find these optimal minimax designs when the order of the polynomial models gets higher. Therefore, we introduce an algorithm for generating the optimal minimax designs numerically, and this algorithm is based on the iterative scheme of Wong (1998).

First we transform our design problems into the optimization problems with the objective function, $\Phi_i$, $i = 1, 2, 3$. Since we require the mass $w$ of the design to satisfy $0 < w \leq 1$ and the supports $x$ to satisfy $x \in \mathcal{X}$, we need to add a penalty function to the objective function $\Phi$ to limit constraints violation. For example when $\mathcal{X} = [-1, 1]$ and there is only one mass $w$, the penalty can be chosen as

$$L = r[(\max[0, x - 1])^2 + (\max[0, -1 - x])^2 + (\max[0, w - 1])^2 + (\max[0, -w])^2],$$

where $r$ is some appropriately chosen positive number. In Wong (1998), Powell’s method and Golden section method is applied to solve our constrained optimization problem, and usually the penalty functions are chosen to be the squared functions. Here Powell’s method (Powell, 1964) is based on the concept of conjugate directions, where direction $S_1$ and $S_2$ are said to be conjugate if $S_1^T HS_2 = 0$ and $H$ is some approximation to the Hessian matrix. A particularly attractive and significant feature of Powell’s method is that, if the objective function is quadratic in each of the $n$ variables, then the function will be minimized in $n$ or fewer conjugate search directions. Please see Vanderplaats (1984) for further details. Therefore, when we give a set of initial design parameters $P_0$
in Powell’s method, the optimizer will output a set of design parameters $P^*$, which may be a local optimum for the objective function. The optimization algorithm is an iterative application of the equation:

$$P_t = P_{t-1} + \alpha_t^* S_t,$$

where $P_t$ is the set of design variables, $S_t$ is the search direction, and $\alpha_t^*$ is a scalar determining the optimal amount of change in $S_t$ at the $t$-th iteration. Here we use the Golden Section method to determine the best $\alpha_t^*$ for given $S_t$. Finally, since designs found may be locally optimal, Theorem 2.1 is used to verify if the design is optimal within the class of all designs.

The generating algorithm in Wong (1998) is described as follow. Since the algorithm is an iterative method, we use $P_t$ to denote the design at the $t$-th iteration. The algorithm for finding an MV-optimal design for $f^T(x) = (1, x, \ldots, x^d)$ on $X = [-1, 1]$ is in the following:

**Step 0.** Set objective function: $\Phi_1$ and variables: $d$, $P_t$, $\lambda(x)$, $Z$, $\varepsilon$.

**Step 1.** Initialize $S_t$ to be coordinate unit vectors, $t = 1, \ldots, 2d + 1$.

**Step 2.** $t \leftarrow 0$.

**Step 3.** $t \leftarrow t + 1$.

**Step 4.** Find $\alpha_t^*$ to minimize $\Phi_1(P_{t-1} + \alpha_t S_t) + rL$ using the Golden Section method.

**Step 5.** $P_t \leftarrow P_{t-1} + \alpha_t^* S_t$

**Step 6.** Check if $t = 2k + 1$. If yes, proceed to **Step 7**. Otherwise, proceed to **Step 3**.

**Step 7.** Create a conjugate direction $S_{t+1} \leftarrow P_t - P_0$

**Step 8.** Find $\alpha_{t+1}^*$ to minimize $\Phi_1(P_t + \alpha_{t+1} S_{t+1}) + rL$ using the Golden Section method.

**Step 9.** $\pi^* \leftarrow P_t + \alpha_{t+1}^* S_{t+1}$

**Step 10.** Check if $\pi^*$ a local minimum by Powell’s method($\varepsilon_p = 10^{-5}$). If yes, proceed to **Step 11**. Otherwise, let $S_t \leftarrow S_{t+1}$, $t = 1, \ldots, 2k + 1$ and $P_0 = \pi^*$, then proceed to **Step 3**.

**Step 11.** Check if $\pi^*$ a global minimum.

**Step 12.** Find $u_1, u_2, \ldots$ in $A_1(\pi^*)$ to minimize $-d(z, \pi^*)$ using the Golden Section method.
Step 13. Find \( \mu^* \) to minimize \( c_1(x^*, \mu, \pi^*) \) using Powell’s method \((x^*: \text{any support point of } \pi^*)\).

Step 14. Check if \( c_1(x^*, \mu^*, \pi^*) \leq \varepsilon \) for all \( x \in [-1, 1] \). If yes, stop and conclude that \( \pi^* \) is MV-optimal for the given tolerance level \( \varepsilon = 10^{-3} \). Otherwise, Randomly select another design \( P_0 \), then proceed to Step 1.

For generating the other two types minimax designs, we need to replace \( A_1 \) to be \( A_2 \) or \( A_3 \); to change the objective function \( \Phi_1 \) to be \( \Phi_2 \) or \( \Phi_3 \), and the corresponding equivalence theorem \( c_1 \) to be \( c_2 \) or \( c_3 \).

Here we simply implement the algorithm to find an MV-optimal design for a simple linear model with a given \( \lambda(x) \) and \( Z \). This is a 3-dimensional problem which requires input variables: \( x_1, x_2 \) and \( w_1 \). Suppose we choose \( P_0 = (x_1, x_2, w_1) = (1, -1, 0.5) \). Then the first cycle of minimization will be in the coordinate directions \( S_1 = (1, 0, 0), S_2 = (0, 1, 0) \) and \( S_3 = (0, 0, 1) \) to determine \( \alpha^*_i, i = 1, \ldots, 2d + 1 \), sequentially in accordance to

\[
P_t = P_{t-1} + \alpha^*_t S_t,
\]

where each \( \alpha^*_t \) is selected to minimize

\[
\Phi_1(P_{t-1} + \alpha_t S_t) + rL
\]

over the real line. Then a conjugate direction \( S_4 = \sum_{i=1}^{3} \alpha^*_i S_i \) is formed, and \( \alpha^*_4 \) is determined to yield a new design with parameters \( P^*_4 = P^*_3 = P_3 + \alpha^*_4 S_4 \). Because of the constraints imposed on the design problem \((-1 \leq x_1 < x_2 \leq 1 \) and \( 0 < w_1 \leq 1 \)), our objective is to minimize

\[
\Phi_1 + rL,
\]

where \( r \) is a multiplier and \( L \) is the penalty function defined by

\[
L = \sum_{i=1}^{2} [(\max[0, x_i - 1])^2 + (\max[0, -1 - x_i])^2] + (\max[0, w_1 - 1])^2 + (\max[0, -w_1])^2.
\]

Note that \( L \) does not penalize \( \Phi \) when the solution is in the feasible set for the problem considered here, i.e. \(-1 \leq x_1 < x_2 \leq 1 \) and \( 0 < w_1 \leq 1 \).

3 Optimal minimax designs for simple linear model

In this section, we consider the simple linear model, i.e. \( f^T(x) = (1, x) \), on \( X = [-1, 1] \) and three different types efficiency function \( \lambda(x) \) are chosen. These three efficiency
functions are linear function, \( x + 5 \); concave function, \( \exp(-5x^2) \), and convex function, \( 0.5x^2 + 1 \). Here for the intervals \( Z \), we choose three types. The first one is symmetric interval, \([-0.8, 0.8] \), \([-1, 1] \) and \([-2, 2] \); the second type is the extrapolation interval, \([1, 1.5] \), and the third type of \( Z \) is \([-1, 1.5] \).

### 3.1 Numerically optimal minimax designs

Now the generating algorithm is applied to find the optimal minimax designs numerically. In the generating algorithm, first the starting design is chosen by randomly picking two distinct support points with equal weights \( 1/2 \), and we fix \( r = 5000 \) in the generating algorithm for all cases.

Given different \( \lambda(x) \) and \( Z \), the corresponding MV-optimal, MP-optimal and MD-optimal designs on \( X = [-1, 1] \) are shown in following:

1. \( \lambda(x) = x + 5 \): The first case is that \( \lambda(x) = x + 5 \). To make sure that the numerically found designs are indeed optimal, we need to check the corresponding \( c_i(x, \mu^*, \pi^*) \), \( i = 1, 2, 3 \), to see if \( c_i(x, \mu^*, \pi^*) \leq 0 \), \( \forall x \in X \). For example, for MV-optimal design with \( \lambda(x) = x + 5 \) and \( Z = [-0.8, 0.8] \), the numerical MV-optimal design is \( \pi_{MV} = \begin{cases} -1 & 1 \\ 0.6 & 0.4 \end{cases} \) with \( \mu^* = \begin{pmatrix} -0.8 \\ 0.397 \end{pmatrix} \), and

\[
\max_{z \in Z} d(z, \pi_{MV}) = \Phi_1(\pm 0.8, \pi_{MV}) = 0.3417.
\]

Now

\[
c_1(x, \mu^*, \pi_{MV}) = -0.1247 - 0.0278x + 0.1247x^2 + 0.0278x^3.
\]

From Figure 1, it is clear that \( \pi_{MV} \) is MV-optimal. For the other cases, the figures of \( c_i(x, \mu^*, \pi^*) \)'s are displayed in Appendix A.1.1. The numerically optimal designs for these three criteria are shown in Table 1.

![Figure 1: Plot for \( c_1(x, \mu^*, \pi_{MV}) \)](image-url)

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2. $\lambda(x) = \exp(-5x^2)$ : Wong (1998) has also studied this case. From his results, we assume that the support points of optimal minimax designs are symmetric for saving computing time and cost. That is, in the algorithm, $(x_1, x_2, w_1)$ is replaced with $(x_1, -x_1, w_1)$. To check if numerical designs are optimal, we still plot $c_i(x, \mu^*, \pi^*)$, and all figures of $c_i(x, \mu^*, \pi^*)$’s are shown in Appendix A.1.2. The corresponding numerically optimal minimax designs are in Table 2.

3. $\lambda(x) = 0.5x^2 + 1$ : Here $\lambda(x) = 0.5x^2 + 1$ is a convex function, and the corresponding numerical results are shown in Table 3. We also plot $c_i(x, \mu^*, \pi^*)$ to make sure our numerical results are optimal, and all figures are shown in Appendix A.1.3. From our results, three numerically optimal minimax designs are the same when $Z = [-0.8, 0.8], [-1, 1], [-2, 2], [1, 1.5]$ and $[-1.5, -1]$. According to our numerically optimal designs, we compare the MV-, MP- and MD-
Table 3 : $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\pi_{MV}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MP}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MD}$</th>
<th>$\mu^*$</th>
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<tbody>
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<td>0.5</td>
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<td>$[-1, 1]$</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>0.5</td>
<td>-1</td>
<td>0.5</td>
<td>-1</td>
<td>0.5</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>0.5</td>
<td>-2</td>
<td>0.5</td>
<td>-1</td>
<td>0.5</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>0.833</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>0.833</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>0.167</td>
<td>-1</td>
<td>0.167</td>
<td>-1</td>
<td>0.167</td>
</tr>
<tr>
<td>$[-1.5, -1]$</td>
<td>1</td>
<td>0.167</td>
<td>-1</td>
<td>0.833</td>
<td>1.5</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>0.375</td>
<td>-1</td>
<td>0.333</td>
<td>-1</td>
<td>0.403</td>
</tr>
</tbody>
</table>

Table 4 : $\Phi_1$-relative efficiency for $\lambda(x) = x + 5$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_{MP}$</td>
<td>$\pi_{MD}$</td>
<td>$\pi_{MV}$</td>
<td>$\pi_{MP}$</td>
</tr>
<tr>
<td>$[-0.8, 0.8]$</td>
<td>0.882</td>
<td>0.870</td>
<td>0.966</td>
</tr>
<tr>
<td>$[-1, 1]$</td>
<td>0.890</td>
<td>0.833</td>
<td>0.957</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>0.888</td>
<td>0.595</td>
<td>0.957</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[-1.5, -1]$</td>
<td>0.922</td>
<td>0.733</td>
<td>0.940</td>
</tr>
</tbody>
</table>

optimal designs by their relative efficiencies, and these relative efficiencies are shown in Tables 4, 5 and 6. Here we highlight these results in the following:

- $\lambda(x) = x + 5$ : From Table 4, we get that three optimal minimax designs are the same with $Z = [1, 1.5]$. For the other intervals, the $\Phi_1$-relative efficiencies of $\pi_{MP}$ and $\Phi_2$-relative efficiencies of $\pi_{MV}$ are all larger than 0.88. Therefore, we think that $\pi_{MV}$’s and $\pi_{MP}$’s have the similar performances for simple linear model. However, for $\pi_{MD}$, these corresponding relative efficiencies are low. Especially, when $Z = [-2, 2]$, the $\Phi_1$- and $\Phi_2$-relative efficiencies of $\pi_{MD}$ are 0.595 and 0.631, respectively.

- $\lambda(x) = \exp(-5x^2)$ : As shown in Table 5, $\pi_{MV}$’s are equal to $\pi_{MP}$’s for all $Z$’s. For $\pi_{MD}$, the $\Phi_1$- and $\Phi_2$-relative efficiencies of $\pi_{MD}$ are at least higher than 0.78. That is, three optimal minimax designs are about equal for MV- and MP-criteria. For the $\Phi_3$-relative efficiencies, the corresponding values of $\pi_{MV}$ and $\pi_{MP}$ are all higher than 0.75, except the case of $Z = [-1, 1.5]$. When $Z = [-1, 1.5]$, these two values
Table 5: $\Phi_i$-relative efficiency for $\lambda(x) = \exp(-5x^2)$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8,0.8]$</td>
<td>$1$</td>
<td>$0.912$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-1,1]$</td>
<td>$1$</td>
<td>$0.887$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-2,2]$</td>
<td>$1$</td>
<td>$0.843$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[1,1.5]$</td>
<td>$1$</td>
<td>$0.973$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-1,1.5]$</td>
<td>$1$</td>
<td>$0.789$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Table 6: $\Phi_i$-relative efficiency for $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8,0.8]$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-1,1]$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-2,2]$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[1,1.5]$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-1.5,-1]$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$[-1,1.5]$</td>
<td>$0.962$</td>
<td>$0.779$</td>
<td>$0.949$</td>
</tr>
</tbody>
</table>

are 0.556 for both $\pi_{MV}$ and $\pi_{MP}$.

- $\lambda(x) = 0.5x^2 + 1$: According to Table 6, three optimal minimax designs are the same except the case of $Z = [-1,1.5]$. For $Z = [-1,1.5]$, all these corresponding relative efficiencies are all larger than 0.75. Therefore, we think that three optimal minimax designs have the similar performance.

From Tables 4, 5 and 6, three optimal minimax designs are equivalent when $\lambda(x) = 0.5x^2 + 1$ and $Z = [-0.8,0.8], [-1,1], [-2,2], [1,1.5]$ and $[-1.5,-1]$. Here we conjecture that these three types of optimal minimax designs are the same when $\lambda(x)$ is a convex function, and $Z$ is a symmetric interval or an extrapolation interval. This is denoted as conjecture 1.

**Conjecture 1.** Let design space $\mathcal{X} = [-1,1]$ and $f^T(x) = (1,x)$. Suppose efficiency function is symmetric convex function over $\mathcal{X}$, and $Z$ is a symmetric interval or an extrapolation interval. Then these three types of optimal minimax designs are equivalent.

Thus, two more cases for symmetric convex efficiency functions are studied here. In these two case, efficiency functions are set to be $x^2 + 1$ and $10x^2 + 1$ with $Z = [-1,1]$ and
Table 7 : $\lambda(x) = cx^2 + 1$

<table>
<thead>
<tr>
<th>$c = 1$</th>
<th>$\pi_{MV}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MP}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MD}$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z = [-1, 1]$</td>
<td>1 0.5 1 0.5</td>
<td>1 0.5 1 0.007</td>
<td>1 0.5 1 0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1 0.5 -1 0.5</td>
<td>-1 0.5 -1 0.007</td>
<td>-1 0.5 -1 0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z = [1, 1.5]$</td>
<td>1 0.833 1.5 1</td>
<td>1 0.833 1.5 1</td>
<td>1 0.833 1.5 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1 0.167</td>
<td>-1 0.167</td>
<td>-1 0.167</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c = 10$</td>
<td>$\pi_{MV}$</td>
<td>$\mu^*$</td>
<td>$\pi_{MP}$</td>
<td>$\mu^*$</td>
<td>$\pi_{MD}$</td>
<td>$\mu^*$</td>
</tr>
<tr>
<td>$Z = [-1, 1]$</td>
<td>1 0.5 1 0.5</td>
<td>1 0.5 0 1</td>
<td>1 0.5 1 0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1 0.5 -1 0.5</td>
<td>-1 0.5</td>
<td>-1 0.5 -1 0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z = [1, 1.5]$</td>
<td>1 0.833 1.5 1</td>
<td>1 0.833 1.5 1</td>
<td>1 0.833 1.5 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1 0.167</td>
<td>-1 0.167</td>
<td>-1 0.167</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[1, 1.5]. The numerical results are shown in Table 7. From Table 7, numerically optimal designs of three minimax criteria are still equivalent for $Z = [-1, 1]$ and $[1, 1.5]$.

### 3.2 Symmetric convex efficiency function

In this subsection, we study Conjecture 1 theoretically. From our numerical results, for simple linear model, the efficiency function is set to be symmetric and convex across $X$. Under this assumption, we prove that $\pi_{MV}$ and $\pi_{MD}$ are equivalent for $Z = [-a, a]$, $[1, b]$, $b > 1$, and $[b, -1]$, $b < -1$, and $\pi_{MV}$ ($\pi_{MD}$) also minimizes $\Phi_2$ under some conditions. Here this subsection is divided into two parts according to two types of $Z$’s, symmetric interval and extrapolation interval.

#### 3.2.1 Symmetric interval

Here we consider that $Z$ is a symmetric interval, and will show that MV- and MP-optimal designs are equivalent for symmetric convex efficiency function. First, we want to prove that $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ is an MV-optimal design directly. Thus, we have the following theorem.

**Theorem 3.1.** Let design space $X = [-1, 1]$ and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is convex and symmetric on $Z = [-a, a]$ and $X$. Then $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ is an MV-optimal design.

**Proof:**
Since $\lambda(x)$ is symmetric over $\mathcal{X}$, i.e. $\lambda(1) = \lambda(-1)$, we have

$$M(\pi^*) = \int_{\mathcal{X}} \lambda(x)f(x)f^T(x)\pi^*(dx) = \lambda(1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

Then

$$f^T(z)M^{-1}(\pi^*)f(z) = \frac{1}{\lambda(1)}(1 + z^2).$$

Thus

$$A_1(\pi^*) = \{u \in \mathcal{Z}\mid \Phi_1(u_1, \pi^*) = \max_{z \in \mathcal{Z}} f^T(z)M^{-1}(\pi^*)f(z)\} = \{\pm a\}.$$

To show $\pi^*$ is an MV-optimal design, we want to show that

$$c_1(x, \mu^*, \pi^*) = \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*)\mu^*(du_1) - \Phi_1(u_1, \pi^*) \leq 0, \forall x \in \mathcal{X},$$

where $\mu^* = \begin{bmatrix} -a & a \\ 1-p & p \end{bmatrix}$. Here we have

$$c_1(x, \mu^*, \pi^*) = \frac{\lambda(x)}{\lambda(1)^2}(1 + 2a(2p - 1)x + a^2x^2) - \frac{1}{\lambda(1)}(1 + a^2).$$

When $p = 0.5$,

$$c_1(x, \mu^*, \pi^*) = \frac{\lambda(x)}{\lambda(1)^2}(1 + a^2x^2) - \frac{1}{\lambda(1)}(1 + a^2).$$

Since both $\lambda(x)$ and $(1 + a^2x^2)$ are convex symmetric on $\mathcal{X}$, their maximum extremes lie on $\pm 1$. So we have $\max_{x \in \mathcal{X}} c_1(x, \mu^*, \pi^*) = c_1(\pm 1, \mu^*, \pi^*) = 0$. Therefore, with $\mu^* = \begin{bmatrix} -a & a \\ 0.5 & 0.5 \end{bmatrix}$, we have

$$c_1(x, \mu^*, \pi^*) \leq 0, \forall x \in \mathcal{X}.$$

Here $\pi^*$ is an MV-optimal design.\[\square\]

After showing $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ is an MV-optimal design, we want to prove that $\pi^*$ is also an MD-optimal design.

**Theorem 3.2.** Let design space $\mathcal{X} = [-1, 1]$ and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is convex and symmetric on $\mathcal{Z} = [-\alpha, \alpha]$ and $\mathcal{X}$. Then $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ is an MD-optimal design.

**Proof:**

Under the assumptions which is $\mathcal{X} = [-1, 1]$, $f^T(x) = (1, x)$, and $\lambda(x)$ is convex and symmetric on $\mathcal{Z} = [-\alpha, \alpha]$ and $\mathcal{X}$, $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ is an MV-optimal design by Theorem 3.1 Given $\pi^* = \begin{bmatrix} -1 & 1 \\ 0.5 & 0.5 \end{bmatrix}$, then

$$\Phi_3(u_3, \pi^*) = \max_{z \in \mathcal{Z}} \left\{ \frac{\lambda(z)}{\lambda(1)}(1 + z^2) \right\}. $$

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Since both $\lambda(z)$ and $(1 + z^2)$ are convex symmetric on $Z = [-a, a]$ and $X$, we have

$$A_3(\pi^*) = \{\pm a\} = A_1(\pi^*).$$

Choose $\mu^* = \left\{ \begin{array}{c} -a \\ 0.5 \\ a \\ 0.5 \end{array} \right\}$, and then

$$c_3(x, \mu^*, \pi^*) = \int_{A_3(\pi^*)} \lambda(x)\lambda(u_3)g(x, u_3, \pi^*)\mu^*(du_3) - \Phi_3(u_3, \pi^*)$$

$$= \lambda(a)\lambda(x)g(x, a, \pi^*) \times 0.5 + \lambda(-a)\lambda(x)g(x, -a, \pi^*) \times 0.5 - \Phi_3(u_3, \pi^*)$$

$$= \lambda(a) \int_{A_3(\pi^*)} \lambda(x)g(x, u_3, \pi^*)\mu^*(du_3) - \frac{\lambda(a)}{\lambda(1)}(1 + a^2)$$

$$= \lambda(a) \left[ \int_{A_3(\pi^*)} \lambda(x)g(x, u_3, \pi^*)\mu^*(du_3) - \frac{1}{\lambda(1)}(1 + a^2) \right]$$

$$= \lambda(a) \left[ \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*)\mu^*(du_1) - \Phi_1(u_1, \pi^*) \right]$$

$$= \lambda(a)c_1(x, \mu^*, \pi^*).$$

Since $\lambda(a) > 0$ and $c_1(x, \mu^*, \pi^*) \leq 0$, $\forall x \in X$, we have

$$c_3(x, \mu^*, \pi^*) \leq 0, \forall x \in X.$$

Hence $\pi^*$ is an MD-optimal design. □

From Theorem 3.1 and Theorem 3.2, we have the following corollary for the equivalence of $\pi_{MV}$ and $\pi_{MD}$.

**Corollary 3.3.** Let design space $X = [-1, 1]$ and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is convex and symmetric on $Z = [-a, a]$ and $X$. Then $MV$-optimal design is the same as $MD$-optimal design.

Since $\lambda(x)$ is convex function, $1/\lambda(x)$ could not be convex function any more. So maximum extremes of $\Phi_2(z, \pi^*)$ may not to lie on boundary of $Z$. Here we prove that $\pi^*$ is also an MP-optimal design under three different conditions of $\mu^*$ in the following theorem.

**Theorem 3.4.** Let design space $X = [-1, 1]$ and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is convex and symmetric on $Z = [-a, a]$ and $X$. Then $\pi^* = \left\{ \begin{array}{c} -1 \\ 0.5 \\ 1 \\ 0.5 \end{array} \right\}$ is an MP-optimal design if

(i) $A_2(\pi^*) = \{0\}$ and $\mu^* = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}$.  

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(ii) $A_2(\pi^*) = \{\pm t\}$, where $0 < t \leq a$, and $\mu^* = \begin{pmatrix} -t \\ 0.5 \\ t \end{pmatrix}$.

(iii) $A_2(\pi^*) = \{\pm a, 0\}$ and $\mu^* = \begin{pmatrix} -a \\ p \\ 0 \\ 1-2p \\ a \\ p \end{pmatrix}$, where $0 \leq p \leq 0.5$.

**Proof:**

(i) When $A_2(\pi^*) = \{0\}$, we have $\mu^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that

$$c_2(x, \mu^*, \pi^*) = \left[ \frac{\lambda(x)}{\lambda(1)^2} + \frac{1}{\lambda(0)} \right] - \Phi_2(u_2, \pi^*).$$

Since $\lambda(x)$ is convex and symmetric on $Z$ and $X$, we have

$$\max_{x \in X} c_2(x, \mu^*, \pi^*) = c_2(\pm 1, \mu^*, \pi^*) = 0,$$

i.e. $\mu^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that $c_2(x, \mu^*, \pi^*) \leq 0$, $\forall x \in [-1, 1]$. Thus, $\pi^*$ with condition (i) is an MP-optimal design.

(ii) We have a design $\pi^* = \begin{pmatrix} -1 \\ 0.5 \\ 1 \end{pmatrix}$, then $\Phi_2(u_2, \pi^*) = \max_{z \in Z} \left\{ f^T(z)M^{-1}(\pi^*) f(z) + 1/\lambda(z) \right\}$. Since $\lambda(x)$ is symmetric on $Z$ and $X$, we get

$$\Phi_2(u_2, \pi^*) = \max_{z \in Z} \left\{ \frac{1}{\lambda(1)}(1 + z^2) + \frac{1}{\lambda(z)} \right\}.$$

When $A_2(\pi^*) = \{\pm t\}$ and $\mu^* = \begin{pmatrix} -t \\ 0.5 \\ t \end{pmatrix}$, we have

$$c_2(x, \mu^*, \pi^*) = \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 + tx^2) + \frac{1}{\lambda(t)} \right] 0.5 + \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 - tx^2) + \frac{1}{\lambda(-t)} \right] 0.5 - \Phi_2(u_2, \pi^*)$$

$$= \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 + t^2x^2) + \frac{1}{\lambda(t)} \right] - \left[ \frac{1}{\lambda(1)}(1 + t^2) + \frac{1}{\lambda(t)} \right].$$

Since $\lambda(x)$ and $(1 + t^2x^2)$ are convex and symmetric on $Z$ and $X$, we have

$$\max_{x \in X} c_2(x, \mu^*, \pi^*) = c_2(\pm 1, \mu^*, \pi^*) = 0.$$

Thus, $\pi^*$ with condition (ii) is an MP-optimal design.

(iii) When $A_2(\pi^*) = \{\pm a, 0\}$ and $\mu^* = \begin{pmatrix} -a \\ p \\ 0 \\ 1-2p \\ a \\ p \end{pmatrix}$, where $0 \leq p \leq 0.5$, we have

$$c_2(x, \mu^*, \pi^*) = \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 + ax^2) + \frac{1}{\lambda(a)} \right] p + \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 - ax^2) + \frac{1}{\lambda(-a)} \right] p$$

$$+ \left[ \frac{\lambda(x)}{\lambda(1)^2} + \frac{1}{\lambda(0)} \right] (1 - 2p) - \Phi_2(u_2, \pi^*)$$

$$= \left[ \frac{\lambda(x)}{\lambda(1)^2} (1 + a^2x^2) + \frac{1}{\lambda(a)} \right] 2p + \left[ \frac{\lambda(x)}{\lambda(1)^2} + \frac{1}{\lambda(0)} \right] (1 - 2p) - \Phi_2(u_2, \pi^*).$$
First, we consider the proof:

\[
\begin{align*}
\Phi_2(a, \pi^*) &= \Phi_2(0, \pi^*) \quad \text{such that } c_2(x, \mu^*, \pi^*) \leq 0, \quad \forall x \in [-1, 1].
\end{align*}
\]

Thus, we have the following two theorems.

From our numerical results, we find that these three types of \( \mu^* \) in Theorem 3.4 do exist. For example, (1) for the case of \( \lambda(x) = 10x^2 + 1 \) and \( Z = [-1, 1], \mu^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), (2) when \( \lambda(x) = 0.5x^2 + 1, Z = [-1, 1] \), we find \( \mu^* = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} \), and (3) when \( \lambda(x) = 0.5x^2 + 1, Z = [-1, 1], \mu^* \) is \( \begin{bmatrix} -1 \\ 0.07 \end{bmatrix} \).

3.2.2 Extrapolation interval

Here we focus on extrapolation intervals. First when \( Z = [1, b], b > 1, \) or \([b, -1], b < -1\), we prove that \( \pi^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) is MV- and MD-optimal directly, where \( p = \frac{b+1}{2b} \).

Thus, We have the following two theorems.

Theorem 3.5. Let \( Z = [1, b], b > 1, \) or \([b, -1], b < -1, \) and \( f^T(x) = (1, x) \). Suppose efficiency function \( \lambda(x) \) is convex and symmetric on design space \( \mathcal{X} = [-1, 1] \). Then \( \pi^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) is an MV-optimal design, where \( p = \frac{b+1}{2b} \).

Proof:

First, we consider \( Z = [1, b], b > 1. \) Since \( \lambda(x) \) is symmetric, i.e. \( \lambda(1) = \lambda(-1) \), we have

\[
M^{-1}(\pi^*) = \frac{b^2}{\lambda(1)(b^2 - 1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]

Then

\[
d(z, \pi^*) = \frac{b^2}{\lambda(1)(b^2 - 1)} (1 - \frac{2}{b}z + z^2).
\]

Since \( b > 1 \), we get that \( d(z, \pi^*) \) has minimum point at \( \frac{1}{b} \), where \( 0 < \frac{1}{b} < 1 \). Thus, \( A_1(\pi^*) = \{b\} \). To show \( \pi^* \) is an MV-optimal design, we need to show that

\[
c_1(x, \mu^*, \pi^*) = \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*)\mu^*(du_1) - \Phi_1(u_1, \pi^*) \leq 0, \quad \forall x \in \mathcal{X},
\]

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where \( \mu^* = \left\{ \frac{b}{1} \right\} \). Here we have
\[
c_1(x, \mu^*, \pi^*) = \frac{\lambda(x)}{\lambda(1)^2} p^2 x^2 - \frac{b^2}{\lambda(1)}. \]

Since both \( \lambda(x) \) and \( x^2 \) are convex symmetric on \( \mathcal{X} \), their maximum extremes lie on \( \pm 1 \). So we have \( \max_{x \in \mathcal{X}} c_1(x, \mu^*, \pi^*) = c_1(\pm 1, \mu^*, \pi^*) = 0 \). Therefore, with \( \mu^* = \left\{ \frac{b}{1} \right\} \), we have
\[
c_1(x, \mu^*, \pi^*) \leq 0, \quad \forall x \in \mathcal{X}. \]

Hence \( \pi^* \) is an MV-optimal design. Following the same procedure, we can also easy prove that \( \pi^* \) is an MV-optimal design for \( Z = [b, -1], b < -1 \). So this proof is omitted here.\( \square \)

**Theorem 3.6.** Let \( Z = [1, b], b > 1 \), or \( [b, -1], b < -1 \), and \( f^T(x) = (1, x) \). Suppose efficiency function \( \lambda(x) \) is convex and symmetric on design space \( \mathcal{X} = [-1, 1] \). Then \( \pi^* = \left\{ \frac{-1}{1-p} \frac{1}{p} \right\} \) is an MD-optimal design, where \( p = \frac{b+1}{2b} \).

**Proof:**

Under the assumptions which is \( \mathcal{X} = [-1, 1] \) and \( f^T(x) = (1, x) \), and \( \lambda(x) \) is convex and symmetric on \( \mathcal{X} \), \( \pi^* = \left\{ \frac{-1}{1-p} \frac{1}{p} \right\} \), where \( p = \frac{b+1}{2b} \), is an MV-optimal design by Theorem 3.5. Given \( \pi^* = \left\{ \frac{-1}{1-p} \frac{1}{p} \right\} \), we have
\[
\Phi_3(u_3, \pi^*) = \max_{z \in \mathcal{Z}} \left\{ \frac{\lambda(z)b^2}{\lambda(1)(b^2-1)}(1 - \frac{2}{b}z + z^2) \right\}. \]

Since \( \lambda(x) \) is convex and symmetric on \( \mathcal{X} \) and \( A_1(\pi^*) = \{ b \} \), we have \( A_3(\pi^*) = \{ b \} \) and \( \mu^* = \left\{ \frac{b}{1} \right\} \). Then
\[
c_3(x, \mu^*, \pi^*)
= \int_{A_3(\pi^*)} \lambda(x)\lambda(u_3)g(x, u_3, \pi^*) \mu^*(du_3) - \Phi_3(u_3, \pi^*)
= \lambda(b)\lambda(x)g(x, b, \pi^*) - \lambda(b)\Phi_1(b, \pi^*)
= \lambda(b)[\lambda(x)g(x, b, \pi^*) - \Phi_1(b, \pi^*)]
= \lambda(b) \left[ \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*) \mu^*(du_1) - \Phi_1(u_1, \pi^*) \right]
= \lambda(b)c_1(x, \mu^*, \pi^*). \]

Since \( \lambda(b) > 0 \) and \( c_1(x, \mu^*, \pi^*) \leq 0, \forall x \in \mathcal{X} \), it implies
\[
c_3(x, \mu^*, \pi^*) \leq 0, \quad \forall x \in \mathcal{X}. \]

Hence \( \pi^* \) is an MD-optimal design. The proof for \( Z = [b, -1], b < -1 \), is similar and is omitted here.\( \square \)
From Theorem 3.5 and Theorem 3.6, we show that \( \pi_{MV} = \pi_{MD} = \{ -1 \ p \over 1-p \} \) for \( Z = [1, b], b > 1, \) or \( [b, -1], b < -1. \) So we have the following corollary.

**Corollary 3.7.** Let \( Z = [1, b], b > 1, \) or \( [b, -1], b < -1. \) and \( f^T(x) = (1, x). \) Suppose efficiency function \( \lambda(x) \) is convex and symmetric on design space \( \mathcal{X} = [-1, 1]. \) Then MV-optimal design is the same as MD-optimal design.

Since \( \lambda(x) \) is convex function, \( 1/\lambda(x) \) is not convex function. So maximum extremes of \( \Phi_2(z, \pi^*) \) may not lie on the boundary of \( Z. \) Here we prove that \( \pi^* \) is also an MP-optimal design if \( \mu^* = \{ b \over 1 \} \) in the following theorem.

**Theorem 3.8.** Let \( Z = [1, b], b > 1, \) or \( [b, -1], b < -1. \) and \( f^T(x) = (1, x). \) Suppose efficiency function \( \lambda(x) \) is convex and symmetric on design space \( \mathcal{X} = [-1, 1]. \) Then \( \pi^* = \{ -1 \ p \over 1-p \} \) is an MP-optimal design if \( A_2(\pi^*) = \{ b \}, \) where \( p = {b+1 \over 2b}. \)

**Proof:**

When \( A_2(\pi^*) = \{ b \}, \) we have \( \mu^* = \{ b \over 1 \}, \) and

\[
\begin{align*}
c_2(x, \mu^*, \pi^*) &= \int_{A_2(\pi^*)} \lambda(x)\lambda(u_2)g(x, u_2, \pi^*)\mu^*(du_2) - \Phi_2(u_2, \pi^*) \\
&= \left[ \frac{1}{\lambda(b)} + \lambda(x)g(x, b, \pi^*) \right] - \left[ \Phi_1(b, \pi^*) + \frac{1}{\lambda(b)} \right] \\
&= \lambda(x)g(x, b, \pi^*) - \Phi_1(b, \pi^*) \\
&= \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*)\mu^*(du_1) - \Phi_1(u_1, \pi^*) \\
&= c_1(x, \mu^*, \pi^*).
\end{align*}
\]

By Theorem 3.4, we have \( c_1(x, \mu^*, \pi^*) \leq 0, \forall x \in \mathcal{X}. \) Thus,

\[
\max_{x \in \mathcal{X}} c_2(x, \mu^*, \pi^*) = c_2(\pm 1, \mu^*, \pi^*) = 0.
\]

That is \( c_2(x, \mu^*, \pi^*) \leq 0, \forall x \in [-1, 1]. \) Here \( \pi^* \) with \( \mu^* = \{ b \over 1 \} \) is an MP-optimal design.\( \Box \)

## 4 Quadratic and cubic models

In this section, we are interested in quadratic and cubic models, on \( \mathcal{X} = [-1, 1]. \) According to the results in Section 3, we only focus on symmetric convex efficiency
Table 8 : $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\pi_{MV}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MP}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MD}$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8, 0.8]$</td>
<td>1</td>
<td>0.223</td>
<td>0.8</td>
<td>0.256</td>
<td>1</td>
<td>0.204</td>
</tr>
<tr>
<td>0</td>
<td>0.554</td>
<td>0</td>
<td>0.488</td>
<td>0</td>
<td>0.592</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.223</td>
<td>-0.8</td>
<td>0.256</td>
<td>-1</td>
<td>0.204</td>
<td>-0.8</td>
</tr>
<tr>
<td>$[-1, 1]$</td>
<td>1</td>
<td>0.286</td>
<td>1</td>
<td>0.287</td>
<td>1</td>
<td>0.268</td>
</tr>
<tr>
<td>0</td>
<td>0.428</td>
<td>0</td>
<td>0.426</td>
<td>0</td>
<td>0.574</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.286</td>
<td>-1</td>
<td>0.287</td>
<td>-1</td>
<td>0.268</td>
<td>-1</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>1</td>
<td>0.274</td>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>0.274</td>
</tr>
<tr>
<td>0</td>
<td>0.452</td>
<td>0</td>
<td>0.452</td>
<td>0</td>
<td>0.452</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.274</td>
<td>-2</td>
<td>0.5</td>
<td>-1</td>
<td>0.274</td>
<td>-2</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>0.496</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>0.496</td>
</tr>
<tr>
<td>0</td>
<td>0.405</td>
<td>0</td>
<td>0.405</td>
<td>0</td>
<td>0.405</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.099</td>
<td>-1</td>
<td>0.099</td>
<td>-1</td>
<td>0.099</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 9 : $\lambda(x) = \exp(x^2)$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\pi_{MV}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MP}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MD}$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8, 0.8]$</td>
<td>1</td>
<td>0.154</td>
<td>0.8</td>
<td>0.178</td>
<td>1</td>
<td>0.119</td>
</tr>
<tr>
<td>0</td>
<td>0.692</td>
<td>0</td>
<td>0.644</td>
<td>0</td>
<td>0.762</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.154</td>
<td>-0.8</td>
<td>0.178</td>
<td>-1</td>
<td>0.119</td>
<td>-0.8</td>
</tr>
<tr>
<td>$[-1, 1]$</td>
<td>1</td>
<td>0.212</td>
<td>1</td>
<td>0.212</td>
<td>1</td>
<td>0.171</td>
</tr>
<tr>
<td>0</td>
<td>0.576</td>
<td>0</td>
<td>0.576</td>
<td>0</td>
<td>0.658</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.212</td>
<td>-1</td>
<td>0.212</td>
<td>-1</td>
<td>0.171</td>
<td>-1</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>1</td>
<td>0.237</td>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>0.237</td>
</tr>
<tr>
<td>0</td>
<td>0.526</td>
<td>0</td>
<td>0.526</td>
<td>0</td>
<td>0.526</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.237</td>
<td>-2</td>
<td>0.5</td>
<td>-1</td>
<td>0.237</td>
<td>-2</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>0.435</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>0.435</td>
</tr>
<tr>
<td>0</td>
<td>0.478</td>
<td>0</td>
<td>0.478</td>
<td>0</td>
<td>0.478</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0.087</td>
<td>-1</td>
<td>0.087</td>
<td>-1</td>
<td>0.087</td>
<td>-1</td>
</tr>
</tbody>
</table>

functions, and assume all support points are symmetric. Here for the intervals $Z$, we consider two types, one is symmetric interval, $[-0.8, 0.8]$, $[-1, 1]$ and $[-2, 2]$, and another one is the extrapolation interval, $[1, 1.5]$. First these numerically optimal minimax designs are found by the generating algorithm in Wong (1998), and then we would like to see whether Conjecture 1 is held or not. To make sure our designs are optimal, we still plot $c_i(x, \mu^*, \pi^*)$’s. All figures of $c_i(x, \mu^*, \pi^*)$’s are shown in Appendix A.1.4, A.1.5 and A.1.6.
Table 10: $\Phi_i$-relative efficiency for $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8, 0.8]$</td>
<td>0.933</td>
<td>0.953</td>
<td>0.890</td>
</tr>
<tr>
<td></td>
<td>0.852</td>
<td>0.862</td>
<td>0.830</td>
</tr>
<tr>
<td>$[-1, 1]$</td>
<td>0.939</td>
<td>0.946</td>
<td>0.956</td>
</tr>
<tr>
<td></td>
<td>0.780</td>
<td>0.790</td>
<td>0.802</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11: $\Phi_i$-relative efficiency for $\lambda(x) = \exp(x^2)$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8, 0.8]$</td>
<td>0.806</td>
<td>0.946</td>
<td>0.721</td>
</tr>
<tr>
<td></td>
<td>0.734</td>
<td>0.779</td>
<td>0.580</td>
</tr>
<tr>
<td>$[-1, 1]$</td>
<td>0.807</td>
<td>0.921</td>
<td>0.635</td>
</tr>
<tr>
<td></td>
<td>0.580</td>
<td>0.631</td>
<td>0.512</td>
</tr>
<tr>
<td>$[-2, 2]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[1, 1.5]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

4.1 Quadratic model

Here $f^T(x) = (1, x, x^2)$ and the design space $\mathcal{X}$ is $[-1, 1]$. In this subsection, there are two symmetric convex efficiency functions considered here. One is $\lambda(x) = 0.5x^2 + 1$, which has been studied in Section 3, and the other one is $\lambda(x) = \exp(x^2)$. Our numerically optimal minimax designs are shown in Tables 8 and 9. Without respect to $\lambda(x)$’s, we get that three numerically optimal minimax designs are the same when $Z$ is a larger symmetric interval, $[-2, 2]$, and an extrapolation interval, $[1, 1.5]$. However, when symmetric intervals are small, these three optimal designs are different to each other. Therefore, we compare these designs by their $\Phi_i$-relative efficiencies, $i = 1, 2, 3$. These relative efficiencies are shown in Tables 10 and 11. From these tables, when $\lambda(x) = 0.5x^2 + 1$, these relative efficiencies are all high. That is, these three types of optimal designs should have the same performances when $\lambda(x) = 0.5x^2+1$. However, when $\lambda(x) = \exp(x^2)$, the corresponding relative efficiencies are low for small symmetric intervals, $[-0.8, 0.8]$ and $[-1, 1]$.

4.2 Cubic model

For cubic model, $f^T(x) = (1, x, x^2, x^3)$ and $\mathcal{X} = [-1, 1]$. The efficiency function we considered here is also symmetric and convex over $\mathcal{X}$, and $\lambda(x)$ is set to be $0.5x^2 + 1$ which has been studied before. Our numerically optimal minimax designs are shown in Table 12. Without respect to $\lambda(x)$’s, here we get that three numerically optimal
Table 12: $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\pi_{MV}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MP}$</th>
<th>$\mu^*$</th>
<th>$\pi_{MD}$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8,0.8]$</td>
<td>1</td>
<td>0.11</td>
<td>0.8</td>
<td>0.171</td>
<td>1</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td>0.48</td>
<td>0.39</td>
<td>0.359</td>
<td>0.329</td>
<td>0.465</td>
<td>0.395</td>
</tr>
<tr>
<td></td>
<td>-0.48</td>
<td>0.39</td>
<td>-0.359</td>
<td>0.329</td>
<td>-0.465</td>
<td>0.395</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>0.11</td>
<td>-0.8</td>
<td>0.171</td>
<td>-1</td>
<td>0.105</td>
</tr>
</tbody>
</table>

| $[-1,1]$  | 1         | 0.21     | 1          | 0.21     | 1          | 0.203    | 1        | 0.195    | 1          | 0.25     | 1        | 0.25     |
|           | 0.45      | 0.29     | 0.444      | 0.29     | 0.45       | 0.297    | 0.423    | 0.305    | 0.482     | 0.25     | 0.481    | 0.25     |
|           | -0.45     | 0.29     | -0.444     | 0.29     | -0.45      | 0.297    | -0.423   | 0.305    | -0.482    | 0.25     | -0.481   | 0.25     |
|           | -1        | 0.21     | -1         | 0.21     | -1         | 0.203    | -1       | 0.195    | -1        | 0.25     | -1       | 0.25     |

| $[-2,2]$  | 1         | 0.188    | 2          | 0.5      | 1          | 0.188    | 2        | 0.5      | 1          | 0.188    | 2        | 0.5      |
|           | 0.512     | 0.312    |            |          | 0.512     | 0.312    |          |          | 0.512     | 0.312    |          |          |
|           | -0.512    | 0.312    |            |          | -0.512    | 0.312    |          |          | -0.512    | 0.312    |          |          |
|           | -1        | 0.188    | -2         | 0.5      | -1        | 0.188    | -2       | 0.5      | -1        | 0.188    | -2       | 0.5      |

| $[1,1.5]$ | 1         | 0.35     | 1.5        | 1        | 1          | 0.35     | 1.5      | 1        | 1          | 0.35     | 1.5      | 1        |
|           | 0.527     | 0.392    |            |          | 0.527     | 0.392    |          |          | 0.527     | 0.392    |          |          |
|           | -0.527    | 0.188    |            |          | -0.527    | 0.188    |          |          | -0.527    | 0.188    |          |          |
|           | -1        | 0.07     |            |          | -1        | 0.07     |          |          | -1        | 0.07     |          |          |

Table 13: $\Phi_i$-relative efficiency for $\lambda(x) = 0.5x^2 + 1$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-0.8,0.8]$</td>
<td>0.952</td>
<td>0.911</td>
<td>0.979</td>
</tr>
<tr>
<td>$[-1,1]$</td>
<td>0.967</td>
<td>0.87</td>
<td>0.977</td>
</tr>
<tr>
<td>$[-2,2]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[1,1.5]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

minimax designs are the same when $Z$ is a larger symmetric interval, $[-2,2]$, and an extrapolation interval, $[1,1.5]$. However, when symmetric intervals are small, these three optimal designs are different to each other. Therefore, we compare these designs by their $\Phi_i$-relative efficiencies, $i = 1, 2, 3$. These relative efficiencies are shown in Table 13. From Table 13, when $\lambda(x) = 0.5x^2 + 1$, these relative efficiencies are all high. That is, these three types of optimal minimax designs should have the same performances.

5 Conclusion

Three minimax criteria are studied in this thesis. Based on the generating algorithm in Wong(1998), given different efficiency functions and intervals $Z$, numerically optimal minimax designs are found for the simple linear, quadratic and cubic models on $\mathcal{X} =$
Then these numerically optimal designs are compared by their relative efficiencies. For the simple linear model, from Tables 4, 5 and 6, no matter what efficiency function is, the performances of $\pi_{MV}$ and $\pi_{MP}$ are similar, because of the higher relative efficiencies. When efficiency function is symmetric and convex over the design space, $\mathcal{X} = [-1, 1]$, we show that $\pi_{MV}$ and $\pi_{MP}$ are equivalent for symmetric intervals and extrapolation intervals, $Z$, and we also show that under certain conditions on $\mu^*$, $\pi_{MP}$ is also the same as $\pi_{MV}$ and $\pi_{MD}$. For the quadratic and cubic models on $[-1, 1]$, we focus on the symmetric convex efficiency functions. For the extrapolation interval, $Z$, these three numerically optimal minimax designs are still the same. However, these three types of optimal minimax designs may not be the same for small symmetric $Z$, but for large symmetric interval, for example, $Z = [-2, 2]$, these numerically optimal designs are still equivalent. From the tables of relative efficiencies, if we must choose one of these three criteria for our experiments, we would suggest $\pi_{MV}$.

The reason for choosing $\pi_{MV}$ is that $\Phi_2$- and $\Phi_3$-relative efficiencies of $\pi_{MV}$ are higher than $\Phi_2$-relative efficiencies of $\pi_{MD}$ and $\Phi_3$-relative efficiencies of $\pi_{MP}$, respectively. That is $\pi_{MV}$ should be a robust design in these optimal criteria.

To prove $\pi_{MV}$ and $\pi_{MD}$ are equivalent, the key point is that the variance function and efficiency function are all convex. Therefore, following similar idea, we can show that for the simple linear model, when $\lambda(x)$ is a concave function, $\pi_{MV}$ is equal to $\pi_{MP}$ when $Z$ is a symmetric interval or an extrapolation interval, because $1/\lambda(x)$ is a convex function. Thus, we have the following theorems, and their proofs are in Appendix A2 and A3.

**Theorem 3.9.** Let $\mathcal{X} = [-1, 1]$ and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is concave and symmetric on $Z = [-a, a]$ and $\mathcal{X}$. Then

1. if $\pi^* = \left\{ \begin{array}{cc} -t & \frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right\}$ with $\mu^* = \left\{ \begin{array}{cc} -a & a \\ 0.5 & 0.5 \end{array} \right\}$ is an MV-optimal design then $\pi^*$ is MP-optimal, where $0 < t \leq 1$.

2. if $\pi^* = \left\{ \begin{array}{cc} -t & \frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right\}$ with $\mu^* = \left\{ \begin{array}{cc} -a & a \\ 0.5 & 0.5 \end{array} \right\}$ is an MP-optimal design then $\pi^*$ is MV-optimal, where $0 < t \leq 1$.

**Theorem 3.10.** Let $Z = [1, b]$, $b > 1$, or $[b, -1]$, $b < -1$, and $f^T(x) = (1, x)$. Suppose efficiency function $\lambda(x)$ is concave and symmetric on design space $\mathcal{X} = [-1, 1]$. Then

1. if $\pi^* = \left\{ \begin{array}{cc} -t & \frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right\}$ is an MV-optimal design then $\pi^*$ is MP-optimal, where $0 < t \leq 1$.

2. if $\pi^* = \left\{ \begin{array}{cc} -t & \frac{1}{2} \\ 0 & \frac{1}{2} \end{array} \right\}$ is an MP-optimal design then $\pi^*$ is MV-optimal, where $0 < t \leq 1$.
From our numerical results, when $Z$ is an extrapolation interval, these three types of optimal minimax designs are all the same for symmetric convex efficiency function. Hence finally we conjecture that for the polynomial model, these three types of optimal minimax designs are equivalent for symmetric convex efficiency functions and extrapolation intervals.
A Appendix

A.1 The Figures of $c_i(x, \mu^*, \pi^*), i = 1, 2, 3$.

A.1.1 The simple liner model with $\lambda(x) = x + 5$.
A.1.2 The simple liner model with $\lambda(x) = \exp(-5x^2)$.

<table>
<thead>
<tr>
<th>$c_1(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_2(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_3(x, \mu^<em>, \pi^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
</tr>
<tr>
<td>$Z = [-1, 1]$</td>
<td>$Z = [-1, 1]$</td>
<td>$Z = [-1, 1]$</td>
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<tr>
<td>$Z = [-2, 2]$</td>
<td>$Z = [-2, 2]$</td>
<td>$Z = [-2, 2]$</td>
</tr>
<tr>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
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<tr>
<td>$Z = [-1, 1.5]$</td>
<td>$Z = [-1, 1.5]$</td>
<td>$Z = [-1, 1.5]$</td>
</tr>
</tbody>
</table>
A.1.3 The simple liner model with \( \lambda(x) = 0.5x^2 + 1 \).

\[
\begin{array}{ccc}
\begin{array}{c}
c_1(x, \mu^*, \pi^*)
\end{array} & \begin{array}{c}
c_2(x, \mu^*, \pi^*)
\end{array} & \begin{array}{c}
c_3(x, \mu^*, \pi^*)
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [-0.8, 0.8]
\end{array} & \begin{array}{c}
Z = [-0.8, 0.8]
\end{array} & \begin{array}{c}
Z = [-0.8, 0.8]
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [-1, 1]
\end{array} & \begin{array}{c}
Z = [-1, 1]
\end{array} & \begin{array}{c}
Z = [-1, 1]
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [-2, 2]
\end{array} & \begin{array}{c}
Z = [-2, 2]
\end{array} & \begin{array}{c}
Z = [-2, 2]
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [1, 1.5]
\end{array} & \begin{array}{c}
Z = [1, 1.5]
\end{array} & \begin{array}{c}
Z = [1, 1.5]
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [-1.5, -1]
\end{array} & \begin{array}{c}
Z = [-1.5, -1]
\end{array} & \begin{array}{c}
Z = [-1.5, -1]
\end{array} \\
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
Z = [-1, 1.5]
\end{array} & \begin{array}{c}
Z = [-1, 1.5]
\end{array} & \begin{array}{c}
Z = [-1, 1.5]
\end{array} \\
\end{array}
\]

26
A.1.4 The quadratic model with \( \lambda(x) = 0.5x^2 + 1 \).

<table>
<thead>
<tr>
<th>( c_1(x, \mu^<em>, \pi^</em>) )</th>
<th>( c_2(x, \mu^<em>, \pi^</em>) )</th>
<th>( c_3(x, \mu^<em>, \pi^</em>) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z = [-0.8, 0.8] )</td>
<td>( Z = [-0.8, 0.8] )</td>
<td>( Z = [-0.8, 0.8] )</td>
</tr>
</tbody>
</table>

\[
Z = [-1, 1] 
\]

\[
Z = [-2, 2] 
\]

\[
Z = [1, 1.5] 
\]

---

27
A.1.5 The quadratic model with $\lambda(x) = \exp(x^2)$.

<table>
<thead>
<tr>
<th>$c_1(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_2(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_3(x, \mu^<em>, \pi^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
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<tr>
<td>$Z = [-1, 1]$</td>
<td>$Z = [-1, 1]$</td>
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<tr>
<td>$Z = [-2, 2]$</td>
<td>$Z = [-2, 2]$</td>
<td>$Z = [-2, 2]$</td>
</tr>
<tr>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
</tr>
</tbody>
</table>
A.1.6 The cubic model with $\lambda(x) = 0.5x^2 + 1$.

<table>
<thead>
<tr>
<th>$c_1(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_2(x, \mu^<em>, \pi^</em>)$</th>
<th>$c_3(x, \mu^<em>, \pi^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
<td>$Z = [-0.8, 0.8]$</td>
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<td>$Z = [-1, 1]$</td>
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<tr>
<td>$Z = [-2, 2]$</td>
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</tr>
<tr>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
<td>$Z = [1, 1.5]$</td>
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<td></td>
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</tr>
</tbody>
</table>
A.2 Proof of Theorem 3.9

(1) Since \( \lambda(x) \) is symmetric over \( \mathcal{X} \), i.e. \( \lambda(t) = \lambda(-t) \), we have

\[
M(\pi^*) = \int_{\mathcal{X}} \lambda(x)f(x)f^T(x)\pi^*(dx) = \lambda(t) \begin{bmatrix} t^2 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then

\[
f^T(z)M^{-1}(\pi^*)f(z) = \frac{1}{\lambda(t)}t^2 + z^2,
\]

and

\[
f^T(z)M^{-1}(\pi^*)f(z) + \frac{1}{\lambda(z)} = \frac{1}{\lambda(t)}t^2 + z^2 + \frac{1}{\lambda(z)}.
\]

Since \( \lambda(z) \) is concave and symmetric on \( Z \), we have \( A_1(\pi^*) = A_2(\pi^*) = \{ \pm a \} \), and

\[
\mu^* = \begin{bmatrix} -a \\ 0.5 \\ 0.5 \\ a \end{bmatrix}.
\]

Thus

\[
c_2(x, \mu^*, \pi^*) = \int_{A_2(\pi^*)} \left[ \frac{1}{\lambda(u_2)} + \lambda(x)g(x, u_2, \pi^*) \right] \mu^*(du_2) - \Phi_2(u_2, \pi^*)
\]

\[
= \int_{A_2(\pi^*)} \lambda(x)g(x, u_2, \pi^*)\mu^*(du_2) + \int_{A_2(\pi^*)} \frac{1}{\lambda(u_2)} \mu^*(du_2) - \Phi_1(u_2, \pi^*) - \frac{1}{\lambda(u_2)}
\]

\[
= \int_{A_2(\pi^*)} \lambda(x)g(x, u_2, \pi^*)\mu^*(du_2) - \Phi_1(u_2, \pi^*)
\]

Since \( \pi^* \) is MV-optimal, it implies

\[
c_2(x, \mu^*, \pi^*) = c_1(x, \mu^*, \pi^*) \leq 0, \quad \forall x \in \mathcal{X}.
\]

Hence \( \pi^* \) is MP-optimal. The proof for (2) of Theorem 3.9 is similar and is omitted here. □
A.3 Proof of Theorem 3.10

(1) Since $\lambda(x)$ is symmetric over $\mathcal{X}$, i.e. $\lambda(t) = \lambda(-t)$, we have

$$M(\pi^*) = \int_{\mathcal{X}} \lambda(x)f(x)f^T(x)\pi^*(dx) = \lambda(t) \begin{bmatrix} t^2 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Then

$$f^T(z)M^{-1}(\pi^*)f(z) = \frac{1}{\lambda(t)}(t^2 + z^2),$$

and

$$f^T(z)M^{-1}(\pi^*)f(z) + \frac{1}{\lambda(z)} = \frac{1}{\lambda(t)}(t^2 + z^2) + \frac{1}{\lambda(z)}.$$

Since $\lambda(z)$ is concave and symmetric on $\mathcal{X}$, we have $A_1(\pi^*) = A_2(\pi^*) = \{b\}$ for $Z = [1, b], \ b > 1$, or $Z = [b, -1], \ b < -1$, and $\mu^* = \begin{bmatrix} b \\ 1 \end{bmatrix}$. Thus

$$c_2(x, \mu^*, \pi^*) = \int_{A_2(\pi^*)} \left[ \frac{1}{\lambda(u_2)} + \lambda(x)g(x, u_2, \pi^*) \right] \mu^*(du_2) - \Phi_2(u_2, \pi^*)$$

$$= \int_{A_2(\pi^*)} \lambda(x)g(x, u_2, \pi^*)\mu^*(du_2) + \int_{A_2(\pi^*)} \frac{1}{\lambda(u_2)}\mu^*(du_2) - \Phi_1(u_2, \pi^*) - \frac{1}{\lambda(u_2)}$$

$$= \int_{A_2(\pi^*)} \lambda(x)g(x, u_2, \pi^*)\mu^*(du_2) - \Phi_1(u_2, \pi^*)$$

$$= \int_{A_1(\pi^*)} \lambda(x)g(x, u_1, \pi^*)\mu^*(du_1) - \Phi_1(u_1, \pi^*)$$

$$= c_1(x, \mu^*, \pi^*).$$

Since $\pi^*$ is MV-optimal, it implies

$$c_2(x, \mu^*, \pi^*) = c_1(x, \mu^*, \pi^*) \leq 0, \ \forall x \in \mathcal{X}.$$ 

Hence $\pi^*$ is MP-optimal. The proof for (2) of Theorem 3.10 is similar and is omitted here. □
References


