Mathematical Statistics Qualifying Test

2010/02/24

1. (10%) A and B agree to meet at a certain place between 1 PM. and 2 PM. Suppose they arrive at the meeting place independently and randomly during the hour. Find the distribution of the length of time that A waits for B. (If B arrives before A, define A's waiting time as 0.)

2. (10%) Let the joint pdf of X_1 and X_2 be

$$f(x_1, x_2; \alpha_1, \alpha_2, \alpha_3) = \frac{1}{B(\alpha_1, \alpha_2, \alpha_3)} \prod_{i=1}^{3} x_i^{\alpha_i - 1},$$

where α_i's> 0, $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 < 1$, and $x_3 = 1 - x_1 - x_2$.

   (a) (5%) Show that

$$B(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)},$$

   where $\Gamma(\cdot)$ is the gamma function.

   (b) (5%) Find the marginal distributions of X_1, X_2 and $X_3 = 1 - X_1 - X_2$.

3. (10%) Let X and Y be independent random variables with $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$, that is, $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$ and $f_Y(y) = \mu e^{-\mu y}$, $y > 0$, $\mu > 0$. Suppose that one can not obtain the observation of X and Y directly. Instead, one observe the random variables Z and W, where

$$Z = \min\{X, Y\} \quad \text{and} \quad W = \begin{cases} 1, & \text{if } Z = X \\ 0, & \text{if } Z = Y \end{cases}.$$

   (a) (5%) Find the joint distribution of Z and W.

   (b) (5%) Prove that Z and W are independent.

4. (15%) Let $X_1, \cdots, X_n$ be a random sample from a population with pdf

$$f_\theta(x) = \frac{1}{\theta}, \ 0 < \theta < x < 2\theta,$$

and $X_{(1)} < \cdots < X_{(n)}$ be the order statistics.

   (a) (5%) Are $X_{(1)}/X_{(n)}$ and $X_{(n)}$ independent random variables?

   (b) (10%) Show that $(X_{(1)}, X_{(n)})$ is minimal sufficient but not complete.

5. (18%) Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random sample from $N(\mu, 1)$ distribution.

   (a) (6%) Find the Cramèr-Rao lower bound (CRLB) of the unbiased estimators of μ^2.

   (b) (6%) Find the maximum likelihood estimator (MLE) and the best unbiased estimator (UMVUE) of μ^2.

   (c) (6%) Derive the variances of the above MLE and UMVUE. Compare them with the CRLB.

6. (17%) Suppose X is one observation from a population with beta(θ,1) pdf.

   (a) (6%) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of the test that rejects H_0 if $X > 1/2$.

   (b) (6%) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

   (c) (5%) Is there a uniformly most powerful test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.

7. (10%) Let $X_1, \cdots, X_n$ be iid Poisson(λ) and assume that λ has a gamma(1,1) prior pdf.

   (a) (5%) Derive the posterior distribution of $2(n+1)\lambda$ given $Y = \sum_{i=1}^{n} X_i$.

   (b) (5%) Find a $1 - \alpha$ credible interval for λ.

8. (10%) For testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, suppose $X_1, \cdots, X_n$ are iid $f(x; \theta)$, $\hat{\theta}$ is the MLE of θ, and $f(x; \theta)$ satisfies the regularity conditions. Prove that under H_0,

$$-2 \log \lambda(\mathbf{X}) \to \chi_1^2 \quad \text{in distribution, as } n \to \infty,$$

   where $\lambda(\mathbf{X})$ is the likelihood ratio test.
   (Hint: Let $\ell(\theta|\mathbf{x})$ denote the log likelihood function and expand $\ell(\theta|\mathbf{x})$ in a 2nd-degree Taylor polynomial around $\hat{\theta}$.)