Mathematical Statistics Qualifying Test

2010/09/15

1. Let $U \sim$ Uniform(0,1) be independent of $W \sim$ Uniform(0,c), where $c > 0$. Let $X = U + W$. Find the distribution function of X. (10%)

2. Let $X_1, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, where μ is **known**.
   - (a) Show that the statistic $T = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a sufficient statistic for σ. (5%)
   - (b) Is T a complete sufficient statistic? (5%)
   - (c) Denote the uniformly minimum variance unbiased estimator (UMVUE) of σ^2 by $\hat{T}$. Find $\hat{T}$. (5%)
   - (d) Let $\hat{T}_c(X_1, \cdots, X_n) = c\sum_{i=1}^n (X_i - \mu)^2$, $c > 0$, be another estimator of σ^2. If one uses the mean squared error to evaluate these estimators, then does the UMVUE always performs better than $\hat{T}_c$? Please verify your answer. (10%)

3. Let $X_1, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, where μ and σ are unknown.
   - (a) Denote the UMVUE of μ and σ^2 by T_1 and T_2, respectively. Find T_1 and T_2. Are T_1 and T_2 independent? [Hint: Using Basu's Theorem.] (10%)
   - (b) In particular, if $\sigma^2 = \mu > 0$, then is T_1 still the UMVUE of μ? Does the variance of T_1 attain the Cramér-Rao lower bound (CRLB)? Please verify your answer. (10%)

4. Let $X_1, \cdots, X_n$ be a random sample from a population with pdf $f(x|\theta) = e^{-(x-\theta)}$, $x \geq \theta$, $\theta \in R$. Let $\hat{\theta}$ denote the MLE of θ.
   - (a) Is $\hat{\theta}$ an asymptotically unbiased estimator of θ? (5%)
   - (b) What is the limiting distribution of $n(\hat{\theta} - \theta)$? (5%)

5. Denote the joint pdf of $\mathbf{X} = (X_1, \cdots, X_n)$ by $f(\mathbf{x}|\theta)$. Let $\pi(\theta)$ denote the prior pdf of θ, $T = T(\mathbf{X})$ be a sufficient statistic for θ with pdf $g(t|\theta)$, and $h(t)$ denote the marginal pdf of T.

(a) Show that if $T(\mathbf{x}) = t$ and $h(t) > 0$, then the posterior pdf of θ, $\pi(\theta|\mathbf{x})$, satisfies

$$\pi(\theta|\mathbf{x}) = \pi(\theta|t) = \frac{g(t|\theta)\pi(\theta)}{h(t)}.$$

[Hint: Using Factorization Theorem for sufficient statistics.] (5%)

(b) What does the above result imply? (5%)

6. Let $X_1, \cdots, X_n$ be a random sample from a Poisson(λ) population. Consider the test of $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$. Please illustrate how to determine the sample size n by the Central Limit Theorem such that a UMP test satisfying $P(\text{reject } H_0|\lambda = 1)$=0.05 and $P(\text{reject } H_0|\lambda = 2)$=0.9? (15%)

7. Let $X_1, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. If σ is known and $\alpha_1 + \alpha_2 \leq \alpha$, show that the shortest confidence interval for μ of the form

$$[\overline{X}_n - z_{1-\alpha_1}\frac{\sigma}{\sqrt{n}}, \overline{X}_n + z_{1-\alpha_2}\frac{\sigma}{\sqrt{n}}]$$

is obtained by taking $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and z_p denotes the pth quantile of the standard normal distribution. (10%)