國立高雄大學統計學研究所
一佰學年度第一學期博士班資格考試

考試日期及時間：100 年 9月 14日 13：00–17：00　　　　　科目：機率論

1. (10 points) A and B play a series of games. Each game is independently won by A with probability p and by B with probability $1-p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series. Find the probability that B is the winner of the series.

2. (15 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of positive random variables such that for each $i=1,2,3$, $\{X_i, X_{i+3}, X_{i+6}, \ldots\}$ are independent and identically distributed (i.i.d.). Furthermore, suppose that $E(|X_1|+|X_2|+|X_3|)<\infty$. Show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \frac{EX_1+EX_2+EX_3}{3} \qquad \text{as} \quad n\to\infty.$$

3. (15 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $P(X_1=1)=P(X_1=-1)=1/2$. Let $S_n=n^{-1/2}\sum_{i=1}^{n} X_i$. Find $\lim\limits_{n\to\infty} P(-1<S_n<1)$.

4. (15 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $P(X_1=1)=P(X_1=-1)=1/2$. Let $T_n=\sum_{i=1}^{n} X_i/2^i$ and $T_n \xrightarrow{a.s.} T$ as $n\to\infty$. Find the characteristic function of T.

5. (15 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. positive random variables with mean 1. Let $S_n=\prod_{i=1}^{n} X_i$, $n\geq 1$. Show that $\{S_n\}_{n\geq 1}$ converges with probability 1.

6. (15 points) A fair coin is tossed repeatedly until three successive heads appear. Find the mean number of tosses required.

7. (15 points) Consider a renewal process for which the lifetimes $X_1, X_2, \ldots$ are independent random variables having the exponential distribution with mean $1/\lambda$. Let $W_0=0$, $W_n=\sum_{i=1}^{n} X_i$, and $N(t)=\max\{n\in\mathbb{N}: W_n\leq t\}$. Find the conditional mean $E[W_5 \mid N(t)=3]$.

$\sim$ End of Paper $\sim$