國立高雄大學統計學研究所

碩士論文

條件常態分佈模型之相容性探討

Compatibility of Conditional Normal Distribution Model

研究生：曾婉芸 撰

指導教授：郭錕霖 博士

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by

Wang-Yun Tseng

Advisor

Kun-Lin Kuo

Institute of Statistics

National University of Kaohsiung

Kaohsiung, Taiwan 811, R.O.C.

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目錄

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指導教授: 郭錕霖
國立高雄大學統計學研究所

學生: 曾婉芸
國立高雄大學統計學研究所

摘要

我們知道多維常態分佈的條件分佈皆是常態分佈，然而在給定條件分佈是常態的情形下，其相對應的聯合分佈不一定存在（此情形稱爲不相容），即使存在（此情形稱爲相容）也不一定是常態分佈。在本研究中，我們將探討條件常態分佈模型是相容的充要條件，同時，在相容的情形下，我們將刻劃所對應的聯合分佈之形式。另一方面，有研究者探討在給定的條件分佈模型是不相容的情形下，試圖計算一個聯合分佈，使得此聯合分佈之條件分佈能盡量地接近所給定的條件分佈；也有研究者提出以不相容的條件分配模型進行缺失資料插補，實證效果也有出人意外的表現。在本次研究中，我們將透過模擬的方式，探討上述兩結果。

關鍵字: 相容性、條件常態分佈、吉氏取樣、僞吉氏分佈

Compatibility of Conditional Normal Distribution Model

Advisor: Kun-Lin Kuo
Institute of Statistics,
National University of Kaohsiung

Student: Wang-Yun Tseng
Institute of Statistics,
National University of Kaohsiung

ABSTRACT

We know that all conditionals of a multivariate normal distribution are normal distributions. However, if the given conditional distributions are normal, there may be no joint distribution having these conditionals (we call this case to be incompatible). Even though there exists a corresponding joint distribution (we call this case to be compatible), it may not be a normal distribution. In this studying, we will provide the necessary and sufficient conditions for compatibility of conditional normal distribution models. In addition, under compatibility, we will characteristize the corresponding joint distribution. On the other hand, when the given conditional distributions are incompatible, some researchers try to compute one joint distribution whose conditional distributions are as close as possible to the given conditional distributions. In addition, some researchers use incompatible conditional distributions to impute the missing data, and the empirical result is well unexpectedly. In this paper, we will discuss the above outcomes.

Keywords and phrases: Compatibility, Conditional Normal Distribution, Gibbs sampler, Pseudo-Gibbs distribution

1 緒論

在傳統的機率課本中，我們被教導若知道隨機向量(X,Y)的聯合機率密度函數$f_{X,Y}(x,y)$，便可由$f_{X,Y}$求得X與Y的邊際機率密度函數$f_X(x)$與$f_Y(y)$，再透過$f_{X,Y}$及f_X和f_Y，即可求得相關的條件機率密度函數$f_{X|Y}(x|y)$與$f_{Y|X}(y|x)$。然而在現實的情形中，我們常不易直接獲得聯合分佈的資訊，反而較容易取得相關條件分佈的特徵，因此有學者探討當已知一組條件常態分佈，如何計算其對應的聯合分佈，見Ahsanullah et al. (1985), Arnold et al. (1988, 1992, 1994), Albajan (1997)。近期，Arnold et al. (2008)提出在二維時，也可透過$X|Y$的條件眾數函數(conditional mode distribution)和$Y|X$的條件分佈函數得到聯合分佈。在相關文獻中，我們發現，當給定條件常態分佈時，其相對應的聯合分佈有可能存在(此情形稱爲相容)或不存在(此情形稱爲不相容)，即使存在，其聯合分佈也不一定是常態分佈。

另一方面，有研究者開始探討在給定的條件分佈模型是不相容的情形下，試圖計算一個聯合分佈，使得此聯合分佈之條件分佈能盡量接近所給定的條件分佈，Van Buuren et al. (2006)提出以不相容的條件分佈模型進行缺失資料插補，也可以得到合理的差補值。然而，我們認爲，即使背後存在一母體，模型也會因估計而導致現有條件分佈不相容，且不相容的程度也會影響到整體應用，所以並非所有不相容條件分佈模型都可以拿來應用。Kuo and Wang (2013)提出給定不相容的條件分佈模型，透過吉氏取樣，經由不同的取樣順序，得到的穩定分佈會有所不同，且稱這個穩定分佈爲僞吉氏分佈(Pseudo-Gibbs distribution)，在此研究中，我們將用模擬的方式，應用僞吉氏分佈的性質，透過二維Kolmogorov-Smirnov(KS)檢定，探討不相容條件常態模型的差異。

在本文中，第二章我們回顧一些條件常態分佈的相關文獻; 第三章我們將Arnold and Press (1989)結果推廣至高維度，探討相容條件常態分佈的充要條件，同時在相容的情形下，我們將刻劃所對應的聯合分佈之形式，並且推導出此條件常態分佈對應的聯合分佈亦服從常態的條件式; 第四章我們透過模擬，比較相同母體在不相容條件常態模型上的差異; 第五章是我們的結論。

2 論文回顧

2.1 常態分佈特徵性質

這一節主要是收集一些文獻中提及二維與多維常態分佈的性質。

在二維情形下，某些型式的條件常態分佈可建構二維常態分佈，Brucker(1979)提出充份條件: 令$X_1|(X_2 = x_2) \sim N(a + bx_2, 1)$和$X_2|(X_1 = x_1) \sim N(c + dx_1, 1)$，若$b = d$且$|d| < 1$，則:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2 \left(\frac{1}{1-d^2} \begin{pmatrix} a+cd \\ c+ad \end{pmatrix}, \frac{1}{1-d^2} \begin{pmatrix} 1 & d \\ d & 1 \end{pmatrix} \right) \tag{1}$$

Fraser and Streit(1980)推廣Brucker(1979)，建立兩個二維常態分佈的性質，敘述定理如下:

定理2.1 (Fraser and Streit, 1980).

二維隨機向量X_1, X_2是二維常態分佈若且爲若X_1是非奇異(non-sigular)的邊際常態分佈和$X_2|(X_1 = x_1) \sim N(c + dx_1, h)$，$\forall x_1 \in R$，其中常數$c, d, h$爲實數且$h > 0$。

定理2.2 (Fraser and Streit, 1980).

二維隨機向量X_1, X_2是二維常態分佈若且爲若$X_1|(X_2 = x_2^0) \sim N(a + bx_2^0, g)$和$X_2|(X_1 = x_1) \sim N(c + dx_1, h)$，$\forall x_1 \in R$，其中，$X_2$的邊際密度函數在$x_2^0$點其值非零(non-zero)和常數$a, b, c, d, g, h$爲實數且$g, h > 0$。如果沒有假設聯合分佈存在，則在給定$X_2 = x_2^0$下的條件變異數$g$，就須有限制式: $g < \frac{h}{d^2}$。

Ahsanullah(1985)將上述所提之定理2.1和定理2.2更一般化，其結果如下:

定理2.3 (Ahsanullah, 1985).

X_1和X_2爲兩獨立同分佈的隨機變數，假設$X_2|(X_1 = x_1) \sim N(\alpha x_1 + \beta, \sigma^2)$和$X_1|(X_2 = x_2) \sim N(\alpha x_2 + \beta, \sigma^2)$，$\forall x_1, x_2 \in R$，則

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2 \left(\frac{1}{1-\alpha} \begin{pmatrix} \beta \\ \beta \end{pmatrix}, \frac{\sigma^2}{1-\alpha^2} \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \right) \tag{2}$$

其中，$|\alpha| < 1$。

Arnold and Pourahamdi(1988)根據他們所提出的定理2.1，得到以下結論:

定理2.4 (Arnold and Pourahamdi, 1988).

X_1和X_2爲兩獨立的隨機變數，假設$X_1|(X_2 = x_2) \sim N(\mu(x_2), \sigma^2(x_2))$和$X_2|(X_1 = x_1) \sim N(\mu(x_1), \sigma^2(x_1))$，其中

$$\mu(x_1) = -\frac{a + bx_1 + cx_1^2}{d + 2cx_1 + fx_1^2}, \quad \sigma^2(x_1) = \frac{1}{d + 2cx_1 + fx_1^2},$$
$$\mu(x_2) = -\frac{a + bx_2 + cx_2^2}{d + 2cx_2 + fx_2^2}, \quad \sigma^2(x_2) = \frac{1}{d + 2cx_2 + fx_2^2},$$

則邊際分佈$f(x_1)$、$f(x_2)$和聯合分佈$f(x_1, x_2)$存在且唯一。

根據Arnold et al. (1988)上述結論，產生令大家感興趣的問題，如果假設X_1和X_2爲兩獨立同分佈的隨機變數，$X_1|(X_2 = x_2) \sim N(\mu(x_2), \sigma^2(x_2))$和$X_2|(X_1 = x_1) \sim N(\mu(x_1), \sigma^2(x_1))$，那麼$\mu(x_1)$、$\mu(x_2)$、$\sigma^2(x_1)$和$\sigma^2(x_2)$該呈現什樣的形式，才能確定此條件分佈是相容的(compatible)呢?關於這一部分，我們在2.2節會再做詳細介紹。

Arnold and Pourahamdi(1988)進一步將定理2.3(Ahsanullah, 1985)擴展至高維度，其定理如下:

定理2.5 (Arnold and Pourahamdi, 1988).

$X_1, X_2, ..., X_n$爲具有一聯合分佈的隨機變數，且$(X_1, X_2, ..., X_{n-1}) \overset{d}{=} (X_2, X_3, ..., X_n)$和$X_n|(X_1 = x_1, X_2 = x_2, ..., X_{n-1} = x_{n-1}) \sim N(\alpha + \sum_{j=1}^{n-1} \beta_j x_j, \sigma^2)$，則$(X_1, X_2, ..., X_n)$服從多維常態分佈。

若$\mathbf{X} = (X_1, X_2, ..., X_k)$是$k$維隨機向量，$\mathbf{X}_{(i)}$表示$\mathbf{X}$向量中沒有第$i$個元素，$\mathbf{X}_{(i,j)}$表示$\mathbf{X}$向量中沒有第$i$個和第$j$個元素，如果$\mathbf{X}$是多維常態分佈，則

(i)$X_i|(\mathbf{X}_{(i)} = \mathbf{x}_{(i)}), \forall \mathbf{x}_{(i)} \in R^{k-1}$是常態分佈;

(ii)$(X_i, X_j)|(\mathbf{X}_{(i,j)} = \mathbf{x}_{(i,j)}), \forall \mathbf{x}_{(i,j)} \in R^{k-2}$是二維常態分佈。

Arnold.et al. (1992)證明出若僅(i)成立，則$\mathbf{X}$並不會是多維常態分佈，而是

$$f_{\mathbf{X}}(\mathbf{x}) \propto \exp\{-\frac{1}{2}\sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\cdots\sum_{i_k=0}^{2}\delta_{i_1,i_2,\ldots,i_k}\prod_{j=1}^{k}x_j^{i_j}\}, \tag{3}$$

其中，參數δ_i需受限制使(3)式在R^k可積分。

同樣的，當僅(ii)成立時，$\mathbf{X}$是否是多維常態分佈?Arnold.et.al (1994)提出以下定理:

定理2.6 (Arnold et al., 1994).

$\mathbf{X}$是多維常態分佈，當(ii)中之條件分佈爲

$$(X_i, X_j)|(\mathbf{X}_{(i,j)} = \mathbf{x}_{(i,j)}) \sim N_2\left(\begin{pmatrix}\mu_i(x_{(i,j)})\\ \mu_j(x_{(i,j)})\end{pmatrix}, \begin{pmatrix}\sigma_{11}(x_{(i,j)}) & \sigma_{12}(x_{(i,j)})\\ \sigma_{21}(x_{(i,j)}) & \sigma_{22}(x_{(i,j)})\end{pmatrix}\right) \tag{4}$$

Albajan and Fidalgo(1997)提出兩點，定理敘述如下:

定理2.7 (Albajan and Fidalgo, 1997).

(1)如果隨機向量

$$Z = \begin{pmatrix}X\\ Y\end{pmatrix} \sim N_n\left(\begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix}, \begin{pmatrix}\Sigma_{XX} & \Sigma_{XY}\\ \Sigma_{YX} & \Sigma_{YY}\end{pmatrix}\right),$$

則$Y|X \sim N(\mu_y + B_0(X-\mu_x), \Sigma_{YY} - B_0\Sigma_{XY})$，其中$B_0$是方程式$\Sigma_{YX} = B\Sigma_{XX}$的解。

(2)如果$X \sim N_r(\mu_x, \Sigma_{XX})$且$Y|X \sim N_s(a + BX, \Sigma_0)$，$n = r + s$，則聯合分佈$(X,Y) \sim N_n(\mu, \Sigma)$，其中，

$$\mu = \begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix} = \begin{pmatrix}\mu_x\\ a + B\mu_x\end{pmatrix},$$

$$\Sigma = \begin{pmatrix}\Sigma_{XX} & \Sigma_{XY}\\ \Sigma_{XY}^T & \Sigma_{YY}\end{pmatrix} = \begin{pmatrix}\Sigma_{XX} & \Sigma_{XX}B^T\\ B\Sigma_{XX} & \Sigma_0 + B\Sigma_{XX}B^T\end{pmatrix}.$$

2.2 相容條件常態分佈之探討:二維與三維

給定兩個條件機率密度函數分別爲$g_{1|2}(x|y)$和$g_{2|1}(y|x)$，若存在一個聯合機率密度函數使得它的條件機率密度函數亦分別爲$f(x|y)$和$f(y|x)$，則此時我們稱這兩個條件分佈是相容的(compatible)。Arnold and Press(1989)提出兩個條件分配是否滿足相容性之檢驗式:

若兩個條件機率密度函數$g_1(x|y)$和$g_2(y|x)$是相容的⇔滿足下列兩個條件:

(1)$N_1 = N_2 = N$，其中$N_1 = \{(x, y)|g_1(x|y) > 0\}$及$N_2 = \{(x, y)|g_2(y|x) > 0\}$。

(2)存在x的函數$u(x)$和y的函數$v(y)$，使得$\dfrac{g_1(x|y)}{g_2(y|x)} = u(x) \cdot v(y)$，對所有的$(x, y) \in N$，且$\int_{-\infty}^{\infty} u(x)dx < \infty$ (或$\int_{-\infty}^{\infty} \frac{1}{v(y)}dy < \infty$)。

在前述情形下，可推得聯合機率密度函數$f(x, y) = g_1(x|y) \cdot \dfrac{u(x)}{\int_{-\infty}^{\infty} u(x)dx}$的條件機率密度函數分別爲$g_1(x|y)$和$g_2(y|x)$。

根據定理2.6(Arnold, 1988)，產生了一些問題，當給定兩個條件分佈形式爲常態時，在什麼情況下這兩個條件分佈是相容的? 亦即兩個條件分佈的平均數及變異數爲何，才能使這兩個條件分佈滿足相容性?若已滿足相容性，此時聯合分佈是否唯一?形式爲何?若此時聯合分佈不爲常態分佈，使否能透過一些限制式，使得聯合分佈爲常態分佈?

Castillo和Galambos(1989)得到以下結果:

X_1和X_2爲兩隨機變數，假設其條件分佈均爲常態分佈，所對應的期望值及變異數爲:

$$E(X_1|X_2 = x_2) = \mu_1(x_2) = -\frac{m_{12}x_2^2 + m_{11}x_2 + m_{10}}{2(m_{22}x_2^2 + m_{21}x_2 + m_{20})}, \tag{5}$$

$$V(X_1|X_2 = x_2) = \sigma_1^2(x_2) = -\frac{1}{2(m_{22}x_2^2 + m_{21}x_2 + m_{20})}, \tag{6}$$

$$E(X_2|X_1 = x_1) = \mu_2(x_1) = -\frac{m_{21}x_1^2 + m_{11}x_1 + m_{01}}{2(m_{22}x_1^2 + m_{12}x_1 + m_{02})}, \tag{7}$$

$$V(X_2|X_1 = x_1) = \sigma_2^2(x_1) = -\frac{1}{2(m_{22}x_1^2 + m_{12}x_1 + m_{02})}, \tag{8}$$

則聯合分佈爲:

$$f(x_1, x_2) = \exp\left\{ \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} \right\}, \tag{9}$$

其中，爲了使(9)式可積分，所以係數m_{ij}必須滿足下列其中一項:

(i)$m_{22} = m_{12} = m_{21} = 0$，$m_{20} < 0$，$m_{02} < 0$，$m_{11}^2 < 4m_{02}m_{20}$，

(ii)$m_{22} < 0$，$m_{12}^2 < 4m_{22}m_{02}$，$m_{21}^2 < 4m_{22}m_{20}$。

因此，如果滿足(i)，則(9)式爲二維常態密度函數，若滿足(ii)，則(9)式爲非常態密度函數但其條件分佈是常態的。蕭惠玲(2010)論文裡有更詳盡的討論。

Gelman和Meng(1991)以比較簡單的方式呈現Castillo et al.,(1989)的結果，並且以圖型舉例説明給定二維條件常態分佈時，聯合分佈的峰態面貌，敘述如下:

X_1, X_2是二維隨機向量，給定條件分佈爲:

$$X_1|(X_2 = x_2) \sim N(\frac{Bx_2 + C_1}{Ax_2^2 + 1}, \frac{1}{Ax_2^2 + 1}), X_2|(X_1 = x_1) \sim N(\frac{Bx_1 + C_2}{Ax_1^2 + 1}, \frac{1}{Ax_1^2 + 1}),$$

則聯合密度函數

$$f(x_1, x_2) \propto \exp(-\frac{1}{2}[Ax_1^2x_2^2 + x_1^2 + x_2^2 - 2Bx_1x_2 - 2C_1x_1 - 2C_1x_1 - 2C_2x_2]),(10)$$

若當$A \geq 0$或$A = 0$和$|B| < 1$，則式子(10)是聯合密度函數。若X_1, X_2是二維常態分佈若且爲若$A = 0$。

(i)考慮$A = 1, B = 0, C_1 = C_2 = C, C = 0$或$C = 1$，則式子(10)圖形爲單峰(unimodal);

(ii)考慮$A = 1, B = 0, C_1 = C_2 = C, C > 2$，則式子(10)圖形爲雙峰(bimodal)。

Arnold et al.,(2000)更進一步的提升到當給定二維條件常態分佈時，聯合分佈的峰態面貌可達到三個峰(trimodal)，結果如下:

考慮迴歸曲線(5)式和(7)式的交叉點，使(5)式代入(7)式可得到X_2的五次方程式，其解可得到1、3或5個根，若是1個根，則圖形爲單峰(unimodal)，若是3個根，則圖形爲雙峰(bimodal)，若是5個根，則圖形爲三個峰(trimodal)。

何霎(2011)將二維條件常態分佈相容性結果，擴展至三維度，其結果如下:

給定三個三維條件均爲常態分佈具有下列性質:

$X|(Y=y,Z=z)\sim N(\mu_1(y,z),\sigma_1^2(y,z))$,$\sigma_1^2(y,z)>0$,$\forall y,z\in R$;

$Y|(X=x,Z=z)\sim N(\mu_2(x,z),\sigma_2^2(x,z))$,$\sigma_2^2(x,z)>0$,$\forall x,z\in R$;

$Z|(X=x,Y=y)\sim N(\mu_3(x,y),\sigma_3^2(x,y))$,$\sigma_3^2(x,y)>0$,$\forall x,y\in R$;

則此三個條件分佈滿足相容性的充分必要條件爲:

存在常數α_{ijk},$0\le i,j,k\le 2$,使得

$$\mu_1(y,z)=(\alpha_{122},\alpha_{121},\alpha_{120},\alpha_{112},\alpha_{111},\alpha_{110},\alpha_{102},\alpha_{101},\alpha_{100})\cdot[(y^2,y,1)\otimes(z^2,z,1)]*\sigma_1^2(y,z)$$

$$\mu_2(x,z)=(\alpha_{212},\alpha_{211},\alpha_{210},\alpha_{112},\alpha_{111},\alpha_{110},\alpha_{012},\alpha_{011},\alpha_{010})\cdot[(x^2,x,1)\otimes(z^2,z,1)]*\sigma_2^2(x,z)$$

$$\mu_1(x,y)=(\alpha_{221},\alpha_{211},\alpha_{201},\alpha_{121},\alpha_{111},\alpha_{101},\alpha_{021},\alpha_{011},\alpha_{001})\cdot[(x^2,x,1)\otimes(y^2,y,1)]*\sigma_3^2(x,y)$$

$$\sigma_1^2(y,z)=\{-2*(\alpha_{222},\alpha_{221},\alpha_{220},\alpha_{212},\alpha_{211},\alpha_{210},\alpha_{202},\alpha_{201},\alpha_{200})\cdot[(y^2,y,1)\otimes(z^2,z,1)]\}^{-1}$$

$$\sigma_2^2(x,z)=\{-2*(\alpha_{222},\alpha_{221},\alpha_{220},\alpha_{122},\alpha_{121},\alpha_{120},\alpha_{022},\alpha_{021},\alpha_{020})\cdot[(x^2,x,1)\otimes(z^2,z,1)]\}^{-1}$$

$$\sigma_3^2(x,y)=\{-2*(\alpha_{222},\alpha_{212},\alpha_{202},\alpha_{122},\alpha_{112},\alpha_{102},\alpha_{022},\alpha_{012},\alpha_{002})\cdot[(x^2,x,1)\otimes(y^2,y,1)]\}^{-1}$$

且$f(x,y,z)\propto\exp(\alpha_{222}x^2y^2z^2+\alpha_{221}x^2y^2z+\alpha_{220}x^2y^2+\alpha_{212}x^2yz^2+\alpha_{211}x^2yz+\alpha_{210}x^2y+\alpha_{202}x^2z^2+\alpha_{201}x^2z+\alpha_{200}x^2+\alpha_{122}xy^2z^2+\alpha_{121}xy^2z+\alpha_{120}xy^2+\alpha_{112}xyz^2+\alpha_{111}xyz+\alpha_{110}xy+\alpha_{102}xz^2+\alpha_{101}xz+\alpha_{100}x+\alpha_{022}y^2z^2+\alpha_{021}y^2z+\alpha_{020}y^2+\alpha_{012}yz^2+\alpha_{011}yz+\alpha_{010}y+\alpha_{002}z^2+\alpha_{001}z)$

是可積分的。

其中,$(a_1,\ldots,a_k)\cdot(b_1,\ldots,b_k)=a_1b_1+\ldots+a_kb_k$;

$(a_1,a_2,a_3)\otimes(b_1,b_2,b_3)=a_1b_1+a_1b_2+a_1b_3+\ldots+a_3b_1+a_3b_2+a_3b_3$。

若此時三個條件分佈滿足相容性,其所對應的聯合分佈亦爲常態分佈之充分必要條件爲:

$\mu_1(y,z)=(\alpha_{110}y+\alpha_{101}z+\alpha_{100})\times\sigma_1^2(y,z)$,$\sigma_1^2(y,z)=(-2\alpha_{200})^{-1}>0$;

$\mu_2(x,z)=(\alpha_{110}y+\alpha_{011}z+\alpha_{010})\times\sigma_2^2(x,z)$,$\sigma_2^2(x,z)=(-2\alpha_{020})^{-1}>0$;

$\mu_3(x,y)=(\alpha_{101}y+\alpha_{011}z+\alpha_{001})\times\sigma_3^2(x,y)$,$\sigma_3^2(x,y)=(-2\alpha_{002})^{-1}>0$;

且$\alpha_{110}^2<4\alpha_{220}\alpha_{020}$和$\alpha_{101}^2<4\alpha_{200}\alpha_{002}$爲滿足聯合分佈可積分條件。

3　高維度條件常態分佈相容性之探討

首先，我們將給予高維度條件分佈族之相容性的充要條件，接著透過此充要條件，我們得到條件常態分佈族之相容性的充要條件，最後，在相容的情形下，由於所對應的聯合分佈不見得是常態分佈，於是我們將給予聯合分佈是常態的充要條件。

定理3.1.

令$f_{i|-i}(x_i|x_{-i})$爲$X_i|(X_{-i}=x_{-i})$的機率密度函數。條件分佈族$\{X_i|(X_{-i}=x_{-i}):1\le i\le n\}$相容的充要條件爲:

(i) 對$2\le k\le n$，存在$W_k(x_{-k})$ 和$U(x_{-1})$，使得$\dfrac{f_{1|-1}(x_1|x_{-1})}{f_{k|-k}(x_k|x_{-k})}=\dfrac{W_k(x_{-k})}{U(x_{-1})}$，

(ii)$\int_{R^{n-1}}U(x_{-1})dx_{-1}<\infty$

證明.

假設相容性成立，於是存在一聯合分佈有$f_{i|-i}$爲其條件機率密度函數，令$U(x_{-1})$爲X_{-1}的邊際機率密度函數，$W_k(x_{-k})$爲X_{-k}的邊際機率密度函數，則(i)與(ii)皆成立。

令$c=\int_{R^{n-1}}U(x_{-1})dx_{-1}$ 和$g(x_{-1})=\frac{U(x_{-1})}{c}$，再令$x=(x_1,\cdots,x_n)$與$g(x)=f_{1|-1}(x_1|x_{-1})\cdot\frac{U(x_{-1})}{c}$，明顯地，$g(x)$是一聯合機率密度函數且$f_{1|-1}$是一條件機率密度函數，對$2\le k\le n$，我們有$g_{k|-k}(x_k|x_{-k})=\frac{g(x)}{\int_{-\infty}^{\infty}g(x)dx_k}=\frac{f_{k|-k}(x_k|x_{-k})\cdot\frac{U_k(x_{-k})}{c}}{\int_{-\infty}^{\infty}\frac{f_{k|-k}(x_k|x_{-k})\cdot W_k(x_{-k})}{c}dx_k}=f_{k|-k}(x_k|x_{-k})$，因此相容性成立。

□

接著，我們討論條件常態分佈族之相容性的充要條件，令$x=(x_1,\cdots,x_n)$，假設給定的條件常態分佈族如下:

$$(\bigstar)\begin{cases}X_1|(X_{-1}=x_{-1})\sim N(\mu_1(x_{-1}),\sigma_1^2(x_{-1}));\\ \vdots\\ X_n|(X_{-n}=x_{-n})\sim N(\mu_n(x_{-n}),\sigma_n^2(x_{-n}));\end{cases}$$

其中$\sigma_1,\cdots,\sigma_n$之函數值皆爲正數，我們想知道這些$\mu_1,\cdots,\mu_n,\sigma_1,\cdots,\sigma_n$ 是什麼函數可使得(★)相容，在回答這個問題前，我們先定義一些特殊符號。

(S1) α是一係數函數定義爲$\alpha(\sum_{j=1}^{k}\prod_{i=1}^{n}x_i^{a_{ij}})=\alpha(x_1^{a_{11}}x_2^{a_{21}}\cdots x_n^{a_{n1}}+x_1^{a_{12}}x_2^{a_{22}}\cdots x_n^{a_{n2}}+\cdots+x_1^{a_{1k}}x_2^{a_{2k}}\cdots x_n^{a_{nk}})\equiv(\alpha_{a_{11}a_{21}\cdots a_{n1}},\alpha_{a_{12}a_{22}\cdots a_{n2}},\cdots,\alpha_{a_{1k}a_{2k}\cdots a_{nk}})$。舉例來說，$\alpha(x_1^2x_2^1x_3^0+$

$x_1x_2^2x_3) = (\alpha_{210}, \alpha_{121})$;

(S2)「$\otimes$」爲克羅內克積(Kronecker product)，假設A是一$n\times m$的矩陣，B是一$p\times q$的矩陣，則$A\otimes B$爲一$np\times mq$的矩陣，且

$$A\otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix},$$

舉例來說，$(x_1^2, x_1, 1)\otimes(x_2^2, x_2, 1) = (x_1^2x_2^2, x_1^2x_2, x_1^2, x_1x_2^2, x_1x_2, x_1, x_2^2, x_2, 1)$;

(S3)「$\cdot$」爲內積，舉例來說，$(x_1, x_2, ...x_k)\cdot(y_1, y_2, ...y_k) = x_1y_1 + x_2y_2 + ... + x_ky_k$。

定理3.2.

(★)相容的充要條件爲存在係數函數α，使得

(1) 對$1\le i\le n$，$\sigma_i^2(x_{-i}) = \{-2[\alpha(x_i^2\otimes_{j\neq i}^n(x_j^2, x_j, 1))]\cdot\otimes_{j\neq i}^n(x_j^2, x_j, 1)\}^{-1}$

$\mu_i(x_{-i}) = \{\alpha(x_i\otimes_{j\neq i}^n(x_j^2, x_j, 1))\cdot\otimes_{j\neq i}^n(x_j^2, x_j, 1)\}\sigma_i^2(x_{-i})$

(2)$\exp\{\alpha(\otimes_{j=1}^n(x_j^2, x_j, 1))\cdot\otimes_{j=1}^n(x_j^2, x_j, 1)\}$是可積分的。

我們以$n=4$爲例，給定:

$X|(Y=y, Z=z, W=w)\sim N(\mu_1(y,z,w), \sigma_1^2(y,z,w))$;

$Y|(X=x, Z=z, W=w)\sim N(\mu_2(x,z,w), \sigma_2^2(x,z,w))$;

$Z|(X=x, Y=y, W=w)\sim N(\mu_3(x,y,w), \sigma_3^2(x,y,w))$;

$W|(X=x, Y=y, Z=z)\sim N(\mu_4(x,y,z), \sigma_4^2(x,y,z))$;

其中$\sigma_1(y,z,w)>0$，$\sigma_2(x,z,w)>0$，$\sigma_3(x,y,w)>0$，$\sigma_4(x,y,z)>0$，$x,y,z,w\in R$，上述條件常態分佈族滿足相容性的充要條件爲存在係數α_{ijkl}，$i,j,k,l\in\{0,1,2\}$，使得:

$$\sigma_1^2(y,z,w) = \{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211}, \alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{2122},\alpha_{2121},$$

$$\alpha_{2120},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{2022},\alpha_{2021},\alpha_{2020}, \alpha_{2012},\alpha_{2011},$$

$$\alpha_{2010},\alpha_{2002},\alpha_{2001},\alpha_{2000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]\}^{-1}$$

$$\sigma_2^2(x,z,w) = \{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211}, \alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{1122},\alpha_{1221},$$

$$\alpha_{1220},\alpha_{1212},\alpha_{1211},\alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{0222},\alpha_{0221},\alpha_{0220}, \alpha_{0212},\alpha_{0211},$$

$\alpha_{0210},\alpha_{0202},\alpha_{0201},\alpha_{0200}) \cdot [(x^2,x,1) \otimes (z^2,z,1) \otimes (w^2,w,1)]\}^{-1}$

$\sigma_3^2(x,y,w) = \{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2122},\alpha_{2121}, \alpha_{2120},\alpha_{2022},\alpha_{2021},\alpha_{2020},\alpha_{1222},\alpha_{1221},$

$\alpha_{1220},\alpha_{1122},\alpha_{1121},\alpha_{1120},\alpha_{1022},\alpha_{1021},\alpha_{1020},\alpha_{0222},\alpha_{0221},\alpha_{0220}, \alpha_{0122},\alpha_{0121},$

$\alpha_{0120},\alpha_{0022},\alpha_{0021},\alpha_{0020}) \cdot [(x^2,x,1) \otimes (y^2,y,1) \otimes (z^2,z,1)]\}^{-1}$

$\sigma_4^2(x,y,z) = \{-2(\alpha_{2222},\alpha_{2212},\alpha_{2202},\alpha_{2122},\alpha_{2112}, \alpha_{2102},\alpha_{2022},\alpha_{2012},\alpha_{2002},\alpha_{1222},\alpha_{1212},$

$\alpha_{1202},\alpha_{1122},\alpha_{1112},\alpha_{1102},\alpha_{1022},\alpha_{1012},\alpha_{1002},\alpha_{0222},\alpha_{0212},\alpha_{0202}, \alpha_{0122},\alpha_{0112},$

$\alpha_{0102},\alpha_{0022},\alpha_{0012},\alpha_{0002}) \cdot [(x^2,x,1) \otimes (y^2,y,1) \otimes (z^2,z,1)]\}^{-1}$

$\mu_1(y,z,w) = (\alpha_{1222},\alpha_{1221},\alpha_{1220},\alpha_{1212},\alpha_{1211}, \alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{1122},\alpha_{1121},\alpha_{1120},$

$\alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{1022},\alpha_{1021},\alpha_{1020}, \alpha_{1012},\alpha_{1011},\alpha_{1010},$

$\alpha_{1002},\alpha_{1001},\alpha_{1000}) \cdot [(y^2,y,1) \otimes (z^2,z,1) \otimes (w^2,w,w)]\sigma_1^2(y,z,w)$

$\mu_2(x,z,w) = (\alpha_{2122},\alpha_{2121},\alpha_{2120},\alpha_{2112},\alpha_{2111}, \alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{1122},\alpha_{1121},\alpha_{1120},$

$\alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{0122},\alpha_{0121},\alpha_{0120}, \alpha_{0112},\alpha_{0111},\alpha_{0110},$

$\alpha_{0102},\alpha_{0101},\alpha_{0100}) \cdot [(x^2,x,1) \otimes (z^2,z,1) \otimes (w^2,w,w)]\sigma_2^2(x,z,w)$

$\mu_3(x,y,w) = (\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2112},\alpha_{2111}, \alpha_{2110},\alpha_{2012},\alpha_{2011},\alpha_{2010},\alpha_{1212},\alpha_{1211},\alpha_{1210},$

$\alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1012},\alpha_{1011},\alpha_{1010},\alpha_{0212},\alpha_{0211},\alpha_{0210}, \alpha_{0112},\alpha_{0111},\alpha_{0110},$

$\alpha_{0012},\alpha_{0011},\alpha_{0010}) \cdot [(x^2,x,1) \otimes (y^2,y,1) \otimes (w^2,w,w)]\sigma_3^2(x,y,w)$

$\mu_4(x,y,z) = (\alpha_{2221},\alpha_{2211},\alpha_{2201},\alpha_{2121},\alpha_{2111}, \alpha_{2101},\alpha_{2021},\alpha_{2011},\alpha_{2001},\alpha_{1221},\alpha_{1211},\alpha_{1201},$

$\alpha_{1121},\alpha_{1111},\alpha_{1101},\alpha_{1021},\alpha_{1011},\alpha_{1001},\alpha_{0221},\alpha_{0211},\alpha_{0201},\alpha_{0121}, \alpha_{0111},\alpha_{0101},$

$\alpha_{0021},\alpha_{0011},\alpha_{0001}) \cdot [(x^2,x,1) \otimes (y^2,y,1) \otimes (w^2,w,w)]\sigma_4^2(x,y,z)$

且$\exp\{\alpha_{2222}x^2y^2z^2w^2 + \alpha_{2221}x^2y^2z^2w + \alpha_{2220}x^2y^2z^2 + \alpha_{2212}x^2y^2zw^2 + \alpha_{2211}x^2y^2zw + \alpha_{2210}x^2y^2z + \alpha_{2202}x^2y^2w^2 + \alpha_{2201}x^2y^2w + \alpha_{2200}x^2y^2 + \alpha_{2122}x^2yz^2w^2 + \alpha_{2121}x^2yz^2w + \alpha_{2120}x^2yz^2 + \alpha_{2112}x^2yzw^2 + \alpha_{2111}x^2yzw + \alpha_{2110}x^2yz + \alpha_{2102}x^2yw^2 + \alpha_{2101}x^2yw + \alpha_{2100}x^2y + \alpha_{2022}x^2z^2w^2 + \alpha_{2021}x^2z^2w + \alpha_{2020}x^2z^2 + \alpha_{2012}x^2zw^2 + \alpha_{2011}x^2zw + \alpha_{2010}x^2z + \alpha_{2002}x^2w^2 + \alpha_{2001}x^2w + \alpha_{2000}x^2 + \alpha_{1222}xy^2z^2w^2 + \alpha_{1221}xy^2z^2w + \alpha_{1220}xy^2z^2 + \alpha_{1212}xy^2zw^2 + \alpha_{1211}xy^2zw + \alpha_{1210}xy^2z + \alpha_{1202}xy^2w^2 + \alpha_{1201}xy^2w + \alpha_{1200}xy^2 + \alpha_{1122}xyz^2w^2 + \alpha_{1121}xyz^2w + \alpha_{1120}xyz^2 + \alpha_{1112}xyzw^2 + \alpha_{1111}xyzw + \alpha_{1110}xyz + \alpha_{1102}xyw^2 + \alpha_{1101}xyw + \alpha_{1100}xy + \alpha_{1022}xz^2w^2 + \alpha_{1021}xz^2w + \alpha_{1020}xz^2 + \alpha_{1012}xzw^2 + \alpha_{1011}xzw + \alpha_{1010}xz + \alpha_{1002}xw^2 + \alpha_{1001}xw + \alpha_{1000}x + \alpha_{0222}y^2z^2w^2 + \alpha_{0221}y^2z^2w + \alpha_{0220}y^2z^2 + \alpha_{0212}y^2zw^2 + \alpha_{0211}y^2zw +$

$\alpha_{0210}y^2z + \alpha_{0202}y^2w^2 + \alpha_{0201}y^2w + \alpha_{0200}y^2 + \alpha_{0122}yz^2w^2 + \alpha_{0121}yz^2w + \alpha_{0120}yz^2 + \alpha_{0112}yzw^2 + \alpha_{0111}yzw + \alpha_{0110}yz + \alpha_{0102}yw^2 + \alpha_{0101}yw + \alpha_{0100}y + \alpha_{0022}z^2w^2 + \alpha_{0012}zw^2 + \alpha_{0021}z^2w + \alpha_{0011}zw + \alpha_{0020}z^2 + \alpha_{0010}z + \alpha_{0002}w^2 + \alpha_{0001}w + +\alpha_{0000}\}$是可積分的。

定理3.2的證明並不容易寫出，當$n = 2$時，見蕭惠玲(2010);當$n = 3$時，見何霙(2011);附錄一中我們證明$n = 4$的情形。

從定理3.2可知:

(i)$\mu_i(x_{-i})$式中，α的第i個下標全爲1，其下標值則是對應變數x的冪次值。

例如:$n = 4$時，$\mu_3(x_{-3})$式中所含的$x_1^2x_4$項係數爲α_{2011}。

(ii)$\sigma_i(x_{-i})$式中，α的第i個下標全爲2，其下標值則是對應變數x的冪次值。

例如:$n = 4$時，$\sigma_3(x_{-3})$式中所含的$x_1^2x_4$項係數爲α_{2021}。

雖然定理3.2已提供了在給定條件常態分佈族滿足相容性的充要條件，不過在使用上仍屬複雜，因此我們可以透過係數判斷的方式，將判斷簡化，我們先考慮$n = 4$的情形，假設給定:

$X|(Y = y, Z = z, W = w) \sim N(\mu_1(y, z, w), \sigma_1^2(y, z, w));$

$Y|(X = x, Z = z, W = w) \sim N(\mu_2(x, z, w), \sigma_2^2(x, z, w));$

$Z|(X = x, Y = y, W = w) \sim N(\mu_3(x, y, w), \sigma_3^2(x, y, w));$

$W|(X = x, Y = y, Z = z) \sim N(\mu_4(x, y, z), \sigma_4^2(x, y, z));$

若$f_{1|2,3,4}(x|y, z, w)$、$f_{2|1,3,4}(y|x, z, w)$、$f_{3|1,2,4}(z|x, y, w)$、$f_{4|1,2,3}(w|x, y, z)$滿足相容性，並且令$f(x, y, z, w)$爲聯合機率密度函數，則存在$U_1(y, z, w), \cdots, U_4(x, y, z)$使得:

$f(x, y, z, w) \propto f(x|y, z, w)U_1(y, z, w);$

$f(x, y, z, w) \propto f(y|x, z, w)U_2(x, z, w);$

$f(x, y, z, w) \propto f(z|x, y, w)U_3(x, y, w);$

$f(x, y, z, w) \propto f(w|x, y, z)U_4(x, y, z);$

將(17)、(18)、(26)、(35)代入可得:

$f(x, y, z, w) \propto \exp\{-\frac{x^2}{2}\frac{1}{\sigma_1^2(y,z,w)} + x\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)} - \frac{1}{2}C_7(z, w)y^2 + C_8(z, w)y\};$

$f(x, y, z, w) \propto \exp\{-\frac{y^2}{2}\frac{1}{\sigma_2^2(x,z,w)} + y\frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)} - \frac{1}{2}C_5(z, w)x^2 + C_6(z, w)x\};$

$f(x,y,z,w) \propto \exp\{-\frac{z^2}{2}\frac{1}{\sigma_3^2(x,y,w)} + z\frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)} - \frac{1}{2}D_5(y,w)x^2 + D_6(y,w)x\}$;
$f(x,y,z,w) \propto \exp\{-\frac{w^2}{2}\frac{1}{\sigma_4^2(x,y,z)} + w\frac{\mu_4(x,y,z)}{\sigma_4^2(x,y,z)} - \frac{1}{2}E_5(y,z)x^2 + E_6(y,z)x\}$;

令$g_1(x,y,z,w) = -\frac{x^2}{2}\frac{1}{\sigma_1^2(y,z,w)} + x\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}$; $g_2(x,y,z,w) = -\frac{y^2}{2}\frac{1}{\sigma_2^2(x,z,w)} + y\frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)}$; $g_3(x,y,z,w) = -\frac{z^2}{2}\frac{1}{\sigma_3^2(x,y,w)} + z\frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)}$; $g_4(x,y,z,w) = -\frac{w^2}{2}\frac{1}{\sigma_4^2(x,y,z)} + w\frac{\mu_4(x,y,z)}{\sigma_4^2(x,y,z)}$，我們可以發現到，$f(x,y,z,w)$爲指數函數，其中包含$x$的項是由$g_1(x,y,z,w)$來，包含$y$的項是由$g_2(x,y,z,w)$來，包含$z$的項是由$g_3(x,y,z,w)$來，包含$w$的項是由$g_4(x,y,z,w)$來，而且，$g_1(x,y,z,w)$、$g_2(x,y,z,w)$、$g_3(x,y,z,w)$、$g_4(x,y,z,w)$中會有同類項，且係數應相同。因此，

$$g_1(x,y,z,w) = -\frac{x^2}{2}\frac{1}{\sigma_1^2(y,z,w)} + x\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}$$
$$= \alpha_{2222}x^2y^2z^2w^2+\alpha_{2221}x^2y^2z^2w+\alpha_{2220}x^2y^2z^2+\alpha_{2212}x^2y^2zw^2+\alpha_{2211}x^2y^2zw+\alpha_{2210}x^2y^2z+\alpha_{2202}x^2y^2w^2+\alpha_{2201}x^2y^2w+\alpha_{2200}x^2y^2+\alpha_{2122}x^2yz^2w^2+\alpha_{2121}x^2yz^2w+\alpha_{2120}x^2yz^2+\alpha_{2112}x^2yzw^2+\alpha_{2111}x^2yzw+\alpha_{2110}x^2yz+\alpha_{2102}x^2yw^2+\alpha_{2101}x^2yw+\alpha_{2100}x^2y+\alpha_{2022}x^2z^2w^2+\alpha_{2021}x^2z^2w+\alpha_{2020}x^2z^2+\alpha_{2012}x^2zw^2+\alpha_{2011}x^2zw+\alpha_{2010}x^2z+\alpha_{2002}x^2w^2+\alpha_{2001}x^2w+\alpha_{2000}x^2+\alpha_{1222}xy^2z^2w^2+\alpha_{1221}xy^2z^2w+\alpha_{1220}xy^2z^2+\alpha_{1212}xy^2zw^2+\alpha_{1211}xy^2zw+\alpha_{1210}xy^2z+\alpha_{1202}xy^2w^2+\alpha_{1201}xy^2w+\alpha_{1200}xy^2+\alpha_{1122}xyz^2w^2+\alpha_{1121}xyz^2w+\alpha_{1120}xyz^2+\alpha_{1112}xyzw^2+\alpha_{1111}xyzw+\alpha_{1110}xyz+\alpha_{1102}xyw^2+\alpha_{1101}xyw+\alpha_{1100}xy+\alpha_{1022}xz^2w^2+\alpha_{1021}xz^2w+\alpha_{1020}xz^2+\alpha_{1012}xzw^2+\alpha_{1011}xzw+\alpha_{1010}xz+\alpha_{1002}xw^2+\alpha_{1001}xw+\alpha_{1000}x;$$

$$g_2(x,y,z,w) = -\frac{y^2}{2}\frac{1}{\sigma_2^2(x,z,w)} + y\frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)}$$
$$= \alpha_{2222}x^2y^2z^2w^2+\alpha_{2221}x^2y^2z^2w+\alpha_{2220}x^2y^2z^2+\alpha_{2212}x^2y^2zw^2+\alpha_{2211}x^2y^2zw+\alpha_{2210}x^2y^2z+\alpha_{2202}x^2y^2w^2+\alpha_{2201}x^2y^2w+\alpha_{2200}x^2y^2+\alpha_{2122}x^2yz^2w^2+\alpha_{2121}x^2yz^2w+\alpha_{2120}x^2yz^2+\alpha_{2112}x^2yzw^2+\alpha_{2111}x^2yzw+\alpha_{2110}x^2yz+\alpha_{2102}x^2yw^2+\alpha_{2101}x^2yw+\alpha_{2100}x^2y+\alpha_{1222}xy^2z^2w^2+\alpha_{1221}xy^2z^2w+\alpha_{1220}xy^2z^2+\alpha_{1212}xy^2zw^2+\alpha_{1211}xy^2zw+\alpha_{1210}xy^2z+\alpha_{1202}xy^2w^2+\alpha_{1201}xy^2w+\alpha_{1200}xy^2+\alpha_{1122}xyz^2w^2+\alpha_{1121}xyz^2w+\alpha_{1120}xyz^2+\alpha_{1112}xyzw^2+\alpha_{1111}xyzw+\alpha_{1110}xyz+\alpha_{1102}xyw^2+\alpha_{1101}xyw+\alpha_{1100}xy+\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0220}y^2z^2+\alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0210}y^2z+\alpha_{0202}y^2w^2+\alpha_{0201}y^2w+\alpha_{0200}y^2+\alpha_{0122}yz^2w^2+\alpha_{0121}yz^2w+\alpha_{0120}yz^2+\alpha_{0112}yzw^2+\alpha_{0111}yzw+\alpha_{0110}yz+\alpha_{0102}yw^2+\alpha_{0101}yw+\alpha_{0100}y;$$

$$g_3(x,y,z,w) = -\frac{z^2}{2}\frac{1}{\sigma_3^2(x,y,w)} + z\frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)}$$

$= \alpha_{2222}x^2y^2z^2w^2+\alpha_{2221}x^2y^2z^2w+\alpha_{2220}x^2y^2z^2+\alpha_{2212}x^2y^2zw^2+\alpha_{2211}x^2y^2zw+\alpha_{2210}x^2y^2z+$
$\alpha_{2122}x^2yz^2w^2+\alpha_{2121}x^2yz^2w+\alpha_{2120}x^2yz^2+\alpha_{2112}x^2yzw^2+\alpha_{2111}x^2yzw+\alpha_{2110}x^2yz+$
$\alpha_{2022}x^2z^2w^2+\alpha_{2021}x^2z^2w+\alpha_{2020}x^2z^2+\alpha_{2012}x^2zw^2+\alpha_{2011}x^2zw+\alpha_{2010}x^2z+\alpha_{1222}xy^2z^2w^2+$
$\alpha_{1221}xy^2z^2w+\alpha_{1220}xy^2z^2+\alpha_{1212}xy^2zw^2+\alpha_{1211}xy^2zw+\alpha_{1210}xy^2z+\alpha_{1122}xyz^2w^2+$
$\alpha_{1121}xyz^2w+\alpha_{1120}xyz^2+\alpha_{1112}xyzw^2+\alpha_{1111}xyzw+\alpha_{1110}xyz+\alpha_{1022}xz^2w^2+\alpha_{1021}xz^2w+$
$\alpha_{1020}xz^2+\alpha_{1012}xzw^2+\alpha_{1011}xzw+\alpha_{1010}xz+\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0220}y^2z^2+$
$\alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0210}y^2z+\alpha_{0122}yz^2w^2+\alpha_{0121}yz^2w+\alpha_{0120}yz^2+\alpha_{0112}yzw^2+$
$\alpha_{0111}yzw+\alpha_{0110}yz+\alpha_{0012}zw^2+\alpha_{0021}z^2w+\alpha_{0011}zw+\alpha_{0020}z^2+\alpha_{0010}z;$

$g_4(x,y,z,w) = -\frac{w^2}{2}\frac{1}{\sigma_4^2(x,y,z)}+w\frac{\mu_4(x,y,z)}{\sigma_4^2(x,y,z)}$
$= \alpha_{2222}x^2y^2z^2w^2+\alpha_{2221}x^2y^2z^2w+\alpha_{2212}x^2y^2zw^2+\alpha_{2211}x^2y^2zw+\alpha_{2202}x^2y^2w^2+$
$\alpha_{2201}x^2y^2w+\alpha_{2122}x^2yz^2w^2+\alpha_{2121}x^2yz^2w+\alpha_{2112}x^2yzw^2+\alpha_{2111}x^2yzw+\alpha_{2102}x^2yw^2+$
$\alpha_{2101}x^2yw+\alpha_{2022}x^2z^2w^2+\alpha_{2021}x^2z^2w+\alpha_{2012}x^2zw^2+\alpha_{2011}x^2zw+\alpha_{2002}x^2w^2+$
$\alpha_{2001}x^2w+\alpha_{1222}xy^2z^2w^2+\alpha_{1221}xy^2z^2w+\alpha_{1212}xy^2zw^2+\alpha_{1211}xy^2zw+\alpha_{1202}xy^2w^2+$
$\alpha_{1201}xy^2w+\alpha_{1122}xyz^2w^2+\alpha_{1121}xyz^2w+\alpha_{1112}xyzw^2+\alpha_{1111}xyzw+\alpha_{1102}xyw^2+$
$\alpha_{1101}xyw+\alpha_{1022}xz^2w^2+\alpha_{1021}xz^2w+\alpha_{1012}xzw^2+\alpha_{1011}xzw+\alpha_{1002}xw^2+\alpha_{1001}xw+$
$\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0202}y^2w^2+\alpha_{0201}y^2w+\alpha_{0122}yz^2w^2+$
$\alpha_{0121}yz^2w+\alpha_{0112}yzw^2+\alpha_{0111}yzw+\alpha_{0102}yw^2+\alpha_{0101}yw+\alpha_{0022}z^2w^2+\alpha_{0012}zw^2+$
$\alpha_{0021}z^2w+\alpha_{0011}zw+\alpha_{0002}w^2+\alpha_{0001}w;$

以及:

$f(x,y,z,w) \propto \frac{1}{\sqrt{2\pi}}\exp\{\alpha_{2222}x^2y^2z^2w^2+\alpha_{2221}x^2y^2z^2w+\alpha_{2220}x^2y^2z^2+\alpha_{2212}x^2y^2zw^2+$
$\alpha_{2211}x^2y^2zw+\alpha_{2210}x^2y^2z+\alpha_{2202}x^2y^2w^2+\alpha_{2201}x^2y^2w+\alpha_{2200}x^2y^2+\alpha_{2122}x^2yz^2w^2+$
$\alpha_{2121}x^2yz^2w+\alpha_{2120}x^2yz^2+\alpha_{2112}x^2yzw^2+\alpha_{2111}x^2yzw+\alpha_{2110}x^2yz+\alpha_{2102}x^2yw^2+$
$\alpha_{2101}x^2yw+\alpha_{2100}x^2y+\alpha_{2022}x^2z^2w^2+\alpha_{2021}x^2z^2w+\alpha_{2020}x^2z^2+\alpha_{2012}x^2zw^2+$
$\alpha_{2011}x^2zw+\alpha_{2010}x^2z+\alpha_{2002}x^2w^2+\alpha_{2001}x^2w+\alpha_{2000}x^2+\alpha_{1222}xy^2z^2w^2+\alpha_{1221}xy^2z^2w+$
$\alpha_{1220}xy^2z^2+\alpha_{1212}xy^2zw^2+\alpha_{1211}xy^2zw+\alpha_{1210}xy^2z+\alpha_{1202}xy^2w^2+\alpha_{1201}xy^2w+$
$\alpha_{1200}xy^2+\alpha_{1122}xyz^2w^2+\alpha_{1121}xyz^2w+\alpha_{1120}xyz^2+\alpha_{1112}xyzw^2+\alpha_{1111}xyzw+$
$\alpha_{1110}xyz+\alpha_{1102}xyw^2+\alpha_{1101}xyw+\alpha_{1100}xy+\alpha_{1022}xz^2w^2+\alpha_{1021}xz^2w+\alpha_{1020}xz^2+$

$\alpha_{1012}xzw^2+\alpha_{1011}xzw+\alpha_{1010}xz+\alpha_{1002}xw^2+\alpha_{1001}xw+\alpha_{1000}x+\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0220}y^2z^2+\alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0210}y^2z+\alpha_{0202}y^2w^2+\alpha_{0201}y^2w+\alpha_{0200}y^2+\alpha_{0122}yz^2w^2+\alpha_{0121}yz^2w+\alpha_{0120}yz^2+\alpha_{0112}yzw^2+\alpha_{0111}yzw+\alpha_{0110}yz+\alpha_{0102}yw^2+\alpha_{0101}yw+\alpha_{0100}y+\alpha_{0022}w^2+\alpha_{0012}zw^2+\alpha_{0021}z^2w+\alpha_{0011}zw+\alpha_{0020}z^2+\alpha_{0010}z+\alpha_{0002}w^2+\alpha_{0001}w\}$;

由上述可知，

(1)$\nabla_{n_1n_2n_3n_4}g_k(x=0,y=0,z=0,w=0)=\alpha_{n_1n_2n_3n_4}$，其中$n_i\in\{0,1,2\}$，
$\nabla_{n_1n_2n_3n_4}=\frac{1}{n_1!n_2!n_3!n_4!}\frac{\partial^{n_1+n_2+n_3+n_4}}{\partial(x^{n_1})\partial(y^{n_2})\partial(z^{n_3})\partial(w^{n_4})}$ 和k滿足$n_k\neq 0$;

(2)$f(x,y,z,w)=\exp(\sum_{(n_1,n_2,n_3,n_4)\in\{0,1,2\}^4}\alpha_{n_1n_2n_3n_4}x^{n_1}y^{n_2}z^{n_3}w^{n_4})$。

例子3.1. (相容情況)

給定條件分佈族:

$X|(Y=y,Z=z,W=w)\sim N(\frac{yzw+2w}{y^2z^2w^2+z^2},\frac{1}{y^2z^2w^2+z^2})$;
$Y|(X=x,Z=z,W=w)\sim N(\frac{xzw+2z}{x^2z^2w^2+w^2},\frac{1}{x^2z^2w^2+w^2})$;
$Z|(X=x,Y=y,W=w)\sim N(\frac{xyw+2y}{x^2y^2w^2+x^2},\frac{1}{x^2y^2w^2+x^2})$;
$W|(X=x,Y=y,Z=z)\sim N(\frac{xyz+2x}{x^2y^2z^2+y^2},\frac{1}{x^2y^2z^2+y^2})$;

解:

令$g_1(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2xw-\frac{1}{2}x^2z^2$;
$g_2(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2yz-\frac{1}{2}y^2w^2$;
$g_3(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2yz-\frac{1}{2}x^2z^2$;
$g_4(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2xw-\frac{1}{2}y^2w^2$;

因此，$x^2y^2z^2w^2$項及$xyzw$項必須同時存在在$g_1(x,y,w,z)$、$g_2(x,y,w,z)$、$g_3(x,y,w,z)$、$g_4(x,y,w,z)$中，且所對應的係數必須相同(如表一)，所以可得:
$\nabla_{2222}g_1(0,0,0,0)=\nabla_{2222}g_2(0,0,0,0)=\nabla_{2222}g_3(0,0,0,0)=\nabla_{2222}g_4(0,0,0,0)=-\frac{1}{2}$，$\nabla_{1111}g_1(0,0,0,0)=\nabla_{1111}g_2(0,0,0,0)=\nabla_{1111}g_3(0,0,0,0)=\nabla_{1111}g_4(0,0,0,0)=1$，係數所對應的均爲$-\frac{1}{2}$和1;

同理，xw項必須同時存在在$g_1(x,y,w,z)$、$g_4(x,y,w,z)$中，所以可得:
$\nabla_{1001}g_1(0,0,0,0)=\nabla_{1001}g_4(0,0,0,0)=2$，係數所對應的均爲2;

同理，xz項必須同時存在在$g_2(x,y,w,z)$、$g_3(x,y,w,z)$中，所以可得:
$\nabla_{0110}g_2(0,0,0,0)=\nabla_{0110}g_3(0,0,0,0)=2$，係數所對應的均爲2;
同理，x^2z^2項必須同時存在在$g_1(x,y,w,z)$、$g_3(x,y,w,z)$中，所以可得:
$\nabla_{2020}g_1(0,0,0,0)=\nabla_{2020}g_3(0,0,0,0)=-\frac{1}{2}$，係數所對應的均爲$-\frac{1}{2}$;
同理，y^2w^2項必須同時存在在$g_2(x,y,w,z)$、$g_4(x,y,w,z)$中，所以可得:
$\nabla_{0202}g_2(0,0,0,0)=\nabla_{0202}g_4(0,0,0,0)=-\frac{1}{2}$，係數所對應的均爲$-\frac{1}{2}$;
因此四個條件分佈滿足相容性。

表一:例子3.1-相容情況

	$x^2y^2z^2w^2$	$xyzw$	xw	yz	x^2z^2	y^2w^2
$g_1(x,y,z,w)$	$-\frac{1}{2}$	1	2		$-\frac{1}{2}$	
$g_2(x,y,z,w)$	$-\frac{1}{2}$	1		2		$-\frac{1}{2}$
$g_3(x,y,z,w)$	$-\frac{1}{2}$	1		2	$-\frac{1}{2}$	
$g_4(x,y,z,w)$	$-\frac{1}{2}$	1	2			$-\frac{1}{2}$

例子3.2. (不相容情況)

給定條件分佈族:
$X|(Y=y,Z=z,W=w)\sim N(\frac{yzw+2y^2w}{y^2z^2w^2+z^2},\frac{1}{y^2z^2w^2+z^2})$;
$Y|(X=x,Z=z,W=w)\sim N(\frac{xzw+2x^2z}{x^2z^2w^2+w^2},\frac{1}{x^2z^2w^2+w^2})$;
$Z|(X=x,Y=y,W=w)\sim N(\frac{xyw+2x^2y}{x^2y^2w^2+x^2},\frac{1}{x^2y^2w^2+x^2})$;
$W|(X=x,Y=y,Z=z)\sim N(\frac{xyz+2x^2y}{x^2y^2z^2+y^2},\frac{1}{x^2y^2z^2+y^2})$;

解:

令$g_1(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2xy^2w-\frac{1}{2}x^2z^2$;
$g_2(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2x^2yz-\frac{1}{2}y^2w^2$;
$g_3(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2x^2yz-\frac{1}{2}x^2z^2$;
$g_4(x,y,z,w)=-\frac{1}{2}x^2y^2z^2w^2+xyzw+2x^2yw-\frac{1}{2}y^2w^2$;
因爲，$2xy^2w$項必須同時存在在$g_1(x,y,w,z)$、$g_2(x,y,w,z)$、$g_4(x,y,w,z)$中，且所對應的係數必須相同，所以，此四個條件分佈不滿足相容性。

接著，我們透過係數判斷的方式，將定理3.2改寫:

推論3.1.

令$x = (x_1, x_2, ..., x_k)$，$N = (n_1, n_2, ..., n_k)$，$n_i \epsilon \{0, 1, 2\}$，給定條件分佈皆爲常態分佈:

$X_i|(X_{-i} = x_{-i}) \sim N(\mu_i(x_{-i}), \sigma_i^2(x_{-i}))$，$\sigma_i^2(x_{-i}) > 0$，$\forall x_{-i} \epsilon R$，

對$1 \leq i \leq n$，$g_i(x) = \frac{\mu_i(x_{-i})}{\sigma_i^2(x_{-i})} x_i - \frac{x_i^2}{2} \cdot \frac{1}{\sigma_i^2(x_{-i})}$，則:

(1)條件常態分佈滿足相容性的充分必要條件爲: $\nabla_N g_{i \backslash i \in \{n_i=0\}}(x = 0)$=$\alpha_N$，其中，

$\nabla_N = \frac{\partial^{n_1+\cdots+n_k}}{\partial(x_1^{n_1})\partial(x_2^{n_2})...\partial(x_k^{n_k})}$;

(2)若條件常態分佈滿足相容性，則聯合分佈$f(x) = \exp(\sum_{N \in \{0,1,2\}^k} \alpha_N \prod_{i=1}^{k} x_i^{n_i})$。

同樣的，從定理3.2可以得知，若給定的條件分佈族皆滿足相容性，其聯合分佈不一定服從常態分佈，下面定理説明何時會產生常態聯合分佈。

定理3.3.

若(★)滿足相容性，則其聯合分佈爲常態分佈若且唯若:

對$1 \leq i \leq n$,

$\sigma_i^2(x_{-i}) = [-2\alpha(x_i^2)]^{-1}$，$\sigma_i^2(x_{-i}) > 0$

$\mu_i(x_{-i}) = [\alpha(x_i(x_{-i}, 1)) \cdot (x_{-i}, 1)]\sigma_i^2(x_{-i})$

此時，聯合分佈密度函數爲$X \sim N(\tilde{\mu}, \Sigma)$，其中，

$$\tilde{\mu} = (\Sigma^{-1})_{ij}\alpha(x_i);$$

$$(\Sigma^{-1})_{ij} \begin{cases} -\alpha(x_i x_j), & i \neq j \\ -2\alpha(x_i^2), & i = j \end{cases}$$

在$n = 4$的情況下，聯合分佈亦爲常態分佈若且唯若:

$\mu_1^2(y, z, w) = (\alpha_{1100}y + \alpha_{1010}z + \alpha_{1001}w + \alpha_{1000})\sigma_1^2(y, z, w)$，$\sigma_1^2(y, z, w) = (-2\alpha_{2000})^{-1}$

$\mu_2^2(x, z, w) = (\alpha_{1100}x + \alpha_{0110}z + \alpha_{0101}w + \alpha_{0100})\sigma_2^2(x, z, w)$，$\sigma_2^2(x, z, w) = (-2\alpha_{0200})^{-1}$

$\mu_3^2(x, y, w) = (\alpha_{1010}x + \alpha_{0110}y + \alpha_{0011}w + \alpha_{0010})\sigma_3^2(x, y, w)$，$\sigma_3^2(x, y, w) = (-2\alpha_{0020})^{-1}$

$\mu_4^2(x, y, z) = (\alpha_{1001}x + \alpha_{0101}y + \alpha_{0011}w + \alpha_{0001})\sigma_4^2(x, y, z)$，$\sigma_4^2(x, y, z) = (-2\alpha_{0002})^{-1}$

此時，聯合密度函數:$(X,Y,Z,W)' \sim N_4(\tilde{\mu},\mathbf{\Sigma})$，其中，

$$\tilde{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \\ \mu_w \end{pmatrix} = \begin{pmatrix} -2\alpha_{2000} & -\alpha_{1100} & -\alpha_{1010} & -\alpha_{1001} \\ -\alpha_{1100} & -2\alpha_{0200} & -\alpha_{0110} & -\alpha_{0101} \\ -\alpha_{1010} & -\alpha_{0110} & -2\alpha_{0020} & -\alpha_{0011} \\ -\alpha_{1001} & -\alpha_{0101} & -\alpha_{0011} & -2\alpha_{0002} \end{pmatrix}^{\mathbf{-1}} \begin{pmatrix} \alpha_{1000} \\ \alpha_{0100} \\ \alpha_{0010} \\ \alpha_{0001} \end{pmatrix};$$

$$\mathbf{\Sigma} = \begin{pmatrix} -2\alpha_{2000} & -\alpha_{1100} & -\alpha_{1010} & -\alpha_{1001} \\ -\alpha_{1100} & -2\alpha_{0200} & -\alpha_{0110} & -\alpha_{0101} \\ -\alpha_{1010} & -\alpha_{0110} & -2\alpha_{0020} & -\alpha_{0011} \\ -\alpha_{1001} & -\alpha_{0101} & -\alpha_{0011} & -2\alpha_{0002} \end{pmatrix}^{-1}.$$

4 不相容條件常態分佈模型之研究

Chen et al.(2013)利用不相容條件模型，試圖計算一個聯合分佈，使得此聯合分佈之條件分佈能盡量接近所給定的條件分佈；另一方面，Van Buuren et al.(2006)應用不相容模型於差補法上，其結果竟得到可接受的差補值。然而，我們認爲在相容與不相容間應有一灰色區域：若理論的條件分佈原屬相容，但模型因估計而導致現有條件分佈不相容者，我們稱之近似相容，當不相容之條件分佈模型若爲近似相容，則探索一聯合分佈來配適這些條件分佈才有意義。因此這一章，我們將對於相容及不相容模型進行測試，並透過僞吉氏分佈（pseudo-Gibbs distribution）的性質，來探討在相同母體下，不同模型間的差異。

4.1 僞吉氏分佈(Pseudo-Gibbs distribution)與二維Kolmogorov-Smirnov檢定

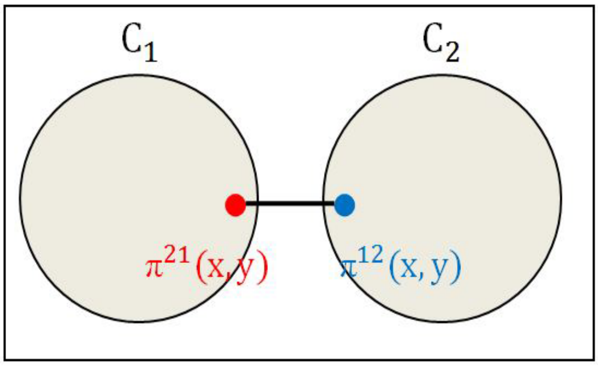
傳統的吉氏取樣(Gibbs sampler)是一種馬可夫鏈蒙地卡羅(Markov Chain Monte Carlo, MCMC)模擬參數的方法，此模擬方法可以讓我們避免掉複雜的計算，即使我們並沒有掌握母體的確切分佈，只需要透過已知的條件分佈，經過足夠次數的疊代後，就能模擬出邊際分佈的觀察值，而這些觀察值可視爲從母體抽出的樣本，因此，我們就能利用所模擬出的樣本來描述眞實母體的相關資訊，其模擬過程如下:

1. 給定任意的初始值 $\mathbf{x^{(0)}} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$;
2. 產生 $x_1^{(1)} \sim f(x_1|x_2^{(0)}, \ldots, x_n^{(0)})$;
   產生 $x_2^{(1)} \sim f(x_2|x_1^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)})$;
   $\vdots$
   產生 $x_n^{(1)} \sim f(x_n|x_1^{(1)}, \ldots, x_{n-1}^{(1)})$，此時已得到$\mathbf{x^{(1)}} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$;
3. 回到步驟2，重複進行m次疊代，則可得$\mathbf{x^{(m)}} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$;

相容的條件分佈透過傳統的吉氏取樣，即使根據不同的取樣順序(scan order)，會產生唯一的穩定分佈(stationary distribution)，相對的，如果給定不相容的條件分佈，透過吉氏取樣，經由不同的取樣順序，收斂到的穩定分佈會不相同，我們則稱這個穩定分佈爲僞吉氏分佈(Pseudo-Gibbs distribution)。在二維情況下，我們定義:若取樣順序爲:$(x_1 \to y_1 \to x_2 \to \cdots \to x_n \to y_n)$，則收斂到的穩定分佈稱$\pi^{12}(x, y)$;若取樣順

序爲:$(y_1 \to x_1 \to y_2 \to \cdots \to y_n \to x_n)$，則收斂到的穩定分佈稱$\pi^{21}(x,y)$。Kuo and Wang(2013)提出在給定兩個聯合分佈集合:$\{C_1 = f(x|y)f_1(y)$和$C_2 = f(y|x)f_2(x)\}$，其中$\pi^{21}(x,y) \in C_1$和$\pi^{12}(x,y) \in C_2$，此時，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$會是集合C_1和C_2之間最接近的位置(如圖一)，我們將應用此性質來比較相同母體在不相容條件常態分佈模型之差異。

圖一:僞吉氏分佈性質之幾何圖形



傳統一維度的Kolmogorov-Smirnov(KS)檢定是一種無母數的方法，用來比較兩個經驗分佈(empirical distribution)，各自的累積分佈函數(cumulative distribution function)間最大絕對距離來衡量分佈是否一致。在一維度時，累積機率函數$\bar{F}_X(x) = 1 - F_X(x)$並無差別，但若擴展到$n$維度，就可以有$2^n - 1$種方式去定義累積分佈函數，所以KS檢定在高維度上的應用，是有一定的困難程度。文獻上，已有一些學者提出KS檢定在二維上的檢定方法(Peacock(1983)、Franceschini and Franceshini(1987))。

在本論文中，我們將採取Fasano and Franceschini(1987)的方法(簡稱FF test)，來檢定兩種分佈是不是來自同一個母體。FF檢定類似一維度的KS檢定，比較兩組資料的累積分佈函數差距，虛無假設$H_0 : F_1(x,y) = F_2(x,y)$，統計量:$D =max\{|F_1(x,y) - F_2(x,y)|\}$，其中，$F_i(\cdot)$爲累積分佈函數，$i = 1,2$。[***本文採用軟體matlab中函數kstest2d***。]

4.2 模擬結果與探討

4.2.1 基本設定

(1)樣本設定:

假設

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

本研究分成三種情況產生1000筆樣本資料，第一種情況:樣本點來自二維常態分佈，其中$\mu_1 = 5$，$\mu_2 = 12$，$\sigma_1^2 = 1$，$\sigma_2^2 = 4$，$\rho = 0.6$，在此種情況下，樣本點皆得到正值;第二種情況:樣本點來自二維常態分佈，其中$\mu_1 = 0$，$\mu_2 = 0$，$\sigma_1^2 = 1$，$\sigma_2^2 = 9$，$\rho = 0.6$，在此種情況下，樣本點正負值均存在，第三種情況:考慮條件分佈$X|(Y = y) \sim N(2/(1+y^2), 1/(1+y^2))$，$Y|(X = x) \sim N(2/(1+x^2), 1/(1+x^2))$，透過傳統吉氏取樣，以burn-in爲20，產生樣本點，根據第三章可知，此時聯合分佈$f(x, y) \propto \exp\{\frac{1}{2}(x^2y^2 + x^2 + y^2 - 4x - 4y)\}$並非常態分佈。

(2)模型設定:

二維常態分佈所對應的條件分佈必爲線性的，因此，兩個條件分佈爲線性和非線性的組合必爲不相容模型，本研究將對於相容及不相容模型進行測試，模型假設如下:

模型I: $\begin{cases} f(x|y) \sim N(b_0 + b_1 y, s_1^2) \\ f(y|x) \sim N(r_0 + r_1 x, s_2^2) \end{cases}$

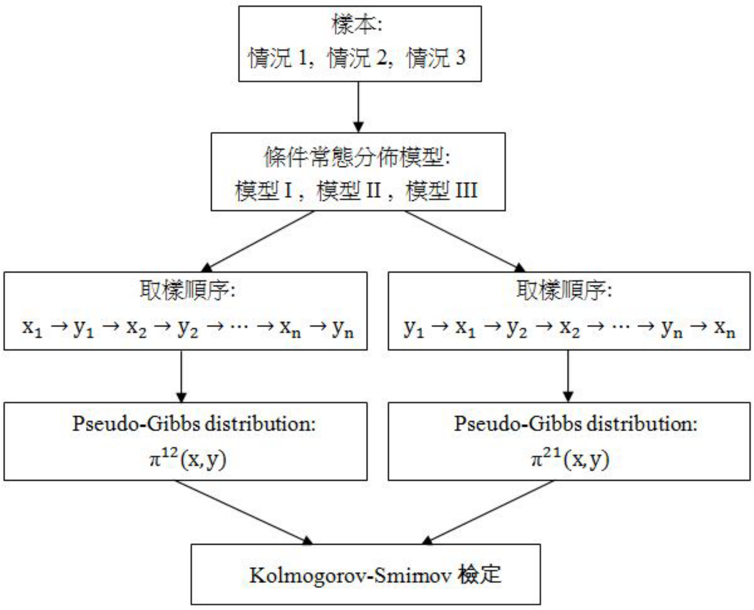
模型II: $\begin{cases} f(x|y) \sim N(b_0 + b_1 y, s_1^2) \\ f(y|x) \sim N(r_0 + r_1 x^2, s_2^2) \end{cases}$

模型III: $\begin{cases} f(x|y) \sim N(b_0 + b_1 y^2, s_1^2) \\ f(y|x) \sim N(r_0 + r_1 x^2, s_2^2) \end{cases}$

4.2.2 模擬步驟

首先，我們將1000筆樣本資料透過迴歸估計參數$(b_0, b_1, r_0, r_1, s_1^2, s_2^2)$，透過相容與不相容模型進行吉氏取樣，其burn-in 都設爲20，根據4.1節偽吉氏分佈性質可知，不同的取樣順序可分別收斂至$\pi^{12}(x,y)$和$\pi^{21}(x,y)$，若相容，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$應該一樣，所以，透過FF test，檢定$H_0 : \pi^{12}(x,y) = \pi^{21}(x,y)$，顯著水準$\alpha = 0.05$，重複100次。圖二爲整個模擬的流程圖。

圖二:流程圖



4.2.3 模擬結果

(一).情況一(樣本來自常態分佈且都爲正值):

由FF檢定的100次結果中(表二):

模型I、模型II和模型III的拒絕次數都在15次內，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$有百分之八十五以上可說兩個分佈一樣，根據圖三(a)、(b)和(c)可知，三種模型的$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 皆近似原母體，所以，由偽吉氏分佈性質可得，在此種情況下，模型I、模型II和模型III皆可說是近似相容，都適合拿來探索一聯合分佈。

(二).情況二(樣本來自常態分佈且正負值均存在):

由FF檢定的100次結果中(表二):

(a) 模型I的拒絕次數爲17次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 有百分之九十可說兩個分佈一樣，根據圖四(a)可知，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 近似原母體，所以，由偽吉氏分佈性質可得，在此種情況下，模型I是近似相容，適合拿來探索一聯合分佈。

(b) 模型II的拒絕次數爲100次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 爲兩個不同分佈，由僞吉氏分佈性質可得，在此種情況下，模型不相容，不適合被應用。

(c) 模型III的拒絕次數爲8次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 有百分之九十以上可說兩個分佈相同，此模型式近似相容。不過由圖四(c)可知，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$與原本體並非一樣，但從表四可得，其平均數和變異數與樣本是接近的。

(三).情況三(樣本非來自常態分佈):

由FF檢定的100次結果中(表二):

(a) 模型I的拒絕次數爲10次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 有百分之九十可說兩個分佈一樣，因此，由僞吉氏分佈性質可得，在此種情況下，模型I還是可以說是近似相容，不過由圖五(a)可知，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$與樣本並非一樣，但由表四可知，其特徵和原母體是近似的。

(b) 模型II的拒絕次數高達82次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 可說是兩個不同分佈，圖五(b)也可觀察到此現象，因此，在此種情況下，模型不適合被應用。

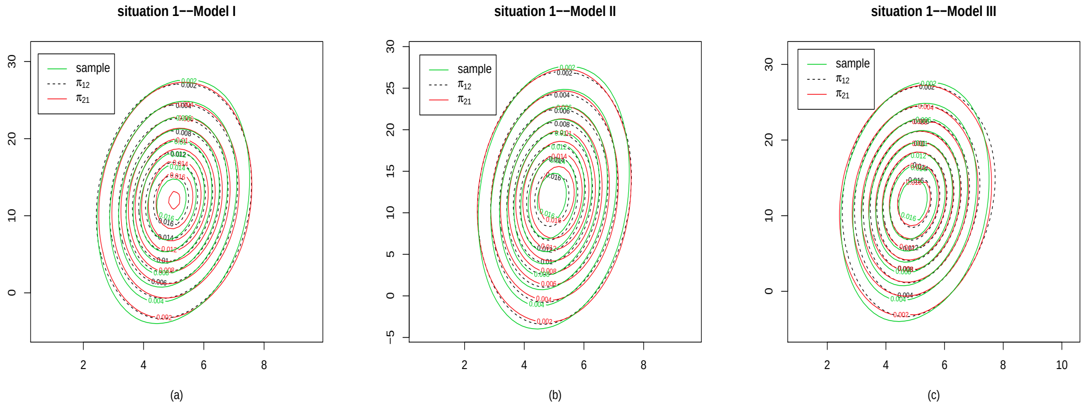
(c) 模型III的拒絕次數爲23次，在信心水準$\alpha = 0.05$下，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$ 有百分之八十左右可說兩個分佈一樣，由僞吉氏分佈性質可得，在此種情況下，模型III還是適用。不過由圖五(c)可知，$\pi^{12}(x,y)$和$\pi^{21}(x,y)$與原母體並非一樣，但從表五可得，其平均數和變異數與樣本是接近的。

表二:100次FF test中，拒絕H_0的次數

	模型I	模型II	模型III
樣本1:常態分佈且爲正値	10	9	14
樣本2:常態分佈且正負值均存在	17	100	8
樣本3: 非常態分佈	10	82	23

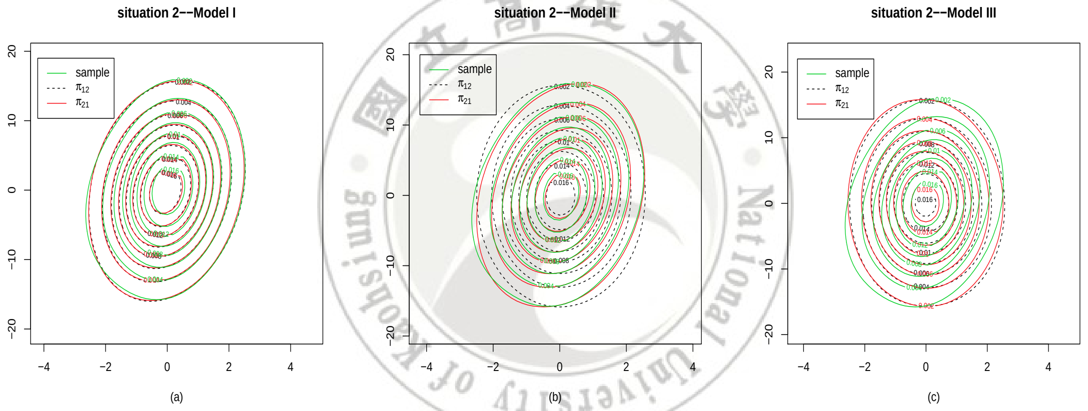
圖三:情況一下，三種模型之樣本、$\pi^{12}(x,y)$和$\pi^{21}(x,y)$等高線圖

(綠色:樣本、虛線:$\pi^{12}(x,y)$、紅色:$\pi^{21}(x,y)$)



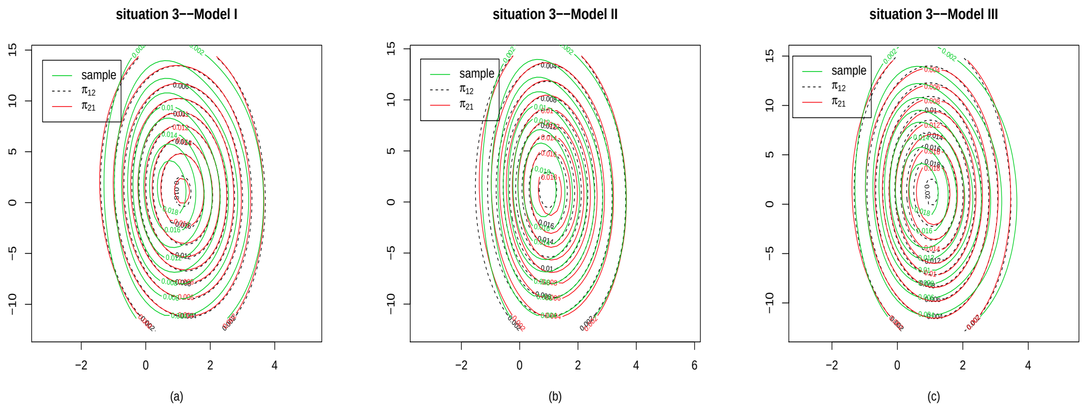
圖四:情況二下，三種模型之樣本、$\pi^{12}(x,y)$和$\pi^{21}(x,y)$等高線圖

(綠色:樣本、虛線:$\pi^{12}(x,y)$、紅色:$\pi^{21}(x,y)$)



圖五:情況三下，三種模型之樣本、$\pi^{12}(x,y)$和$\pi^{21}(x,y)$等高線圖

(綠色:樣本、虛線:$\pi^{12}(x,y)$、紅色:$\pi^{21}(x,y)$)



表三:情況一

		模型I		模型II		模型III	
	樣本	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$
$\hat{\mu_1}$	5.016	4.984	5.057	4.984	5.057	5.070	4.998
$\hat{\mu_2}$	11.991	11.995	12.045	11.995	12.045	12.142	12.024
$\hat{\sigma_1^2}$	1.009	1.052	0.984	1.052	0.984	1.068	1.077
$\hat{\sigma_2^2}$	3.923	3.955	3.883	3.955	3.883	4.268	4.273
$\hat{\rho}$	0.594	0.623	0.596	0.623	0.596	0.624	0.603

表四:情況二

		模型I		模型II		模型III	
	樣本	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$
$\hat{\mu_1}$	0.002	-0.023	-0.034	0.035	0.053	0.024	-0.024
$\hat{\mu_2}$	-0.043	-0.078	-0.039	-0.049	-0.060	-0.022	0.064
$\hat{\sigma_1^2}$	0.983	0.992	0.967	0.991	0.963	1.022	0.995
$\hat{\sigma_2^2}$	8.533	8.757	8.448	8.885	8.941	9.375	8.877
$\hat{\rho}$	0.623	0.611	0.613	0.033	0.647	-0.032	-0.059

表五:情況三

		模型I		模型II		模型III	
	樣本	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$	$\pi^{12}(x,y)$	$\pi^{21}(x,y)$
$\hat{\mu_1}$	1.124	1.119	1.127	1.046	1.113	1.091	1.144
$\hat{\mu_2}$	1.074	1.105	1.086	1.067	1.080	1.170	1.142
$\hat{\sigma_1^2}$	0.916	0.849	0.864	0.921	0.965	0.808	0.835
$\hat{\sigma_2^2}$	0.910	0.915	0.883	0.989	0.894	0.861	0.887
$\hat{\rho}$	-0.613	-0.602	-0.601	-0.353	-0.549	-0.373	-0.302

5 結論

本文主要推導出當給定條件分佈服從常態時，其滿足相容性的充分必要條件。且可透過係數判斷的方式，簡化其相容性之檢驗過程。我們也可得到當給定條件分佈是常態的情形下，其相對應的聯合分佈不一定存在，若存在也不一定是常態分佈。當條件分佈之變異數限制爲正的常數，則可推得到滿足相容性且聯合分佈亦爲常態分佈時之充要條件。

本文也透過電腦模擬的方式，探討三種模型在三種母體下的差異。我們可得到，若母體來自常態分佈且都取正值，透過相容模型(模型I)或不相容模型(模型II、模型III)估計出來的結果，都是近似相容，所以Buuren et al.(2006) 透過不相容模型進行差補法，結果相當不錯，是很合理的。若母體來自常態分佈且正負值均存在或是母體非常態分佈，模型II都不適合拿來配適，此結果也和Drechsler et al.(2008)一致。必須注意到，透過不同取樣順序得到的僞吉氏分佈，即使檢定結果是相同分佈，並不能保證和原母體分佈一樣，但還是可以說明模型是相容或近似相容。反之，若檢定結果爲不同分佈，則模型就是不相容的，不適合拿來應用。

最後，我們希望未來可將近似相容的判斷方式更一般化，給定的兩組條件分佈時，可考慮設立一門檻的方式，直接透過門檻值來判斷是否近似相容。

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附錄一:定理3.2當$n=4$時的證明

根據定理3.1可知，當$n=4$時，考慮隨機變數X,Y,Z,W，則$U(y,z,w)$與$W_2(x,z,w)$進行除法運算時，有些z,w的項可能會相消，同理，$U(y,z,w)$與$W_3(x,y,w)$和$U(y,z,w)$與$W_4(x,y,z)$亦可能會把某些項相消，因此，我們可將定理3.1的充要條件改寫成:

(i) 存在$V_1(x,z,w)$、$V_2(x,y,w)$、$V_3(x,y,z)$、$U_1(y,z,w)$、$U_2(y,z,w)$、$U_3(y,z,w)$、$H_1(z,w)$、$H_2(y,w)$、$H_3(y,z)$、$G_1(y)$、$G_2(z)$、$G_3(w)$，使得:
$\frac{f_{1|2,3,4}(x|y,z,w)}{f_{2|1,3,4}(y|x,z,w)}=\frac{V_1(x,z,w)}{U_1(y,z,w)}$、$\frac{f_{1|2,3,4}(x|y,z,w)}{f_{3|1,2,4}(z|x,y,w)}=\frac{V_2(x,y,w)}{U_2(y,z,w)}$、
$\frac{f_{1|2,3,4}(x|y,z,w)}{f_{4|1,2,3}(w|x,y,z)}=\frac{V_3(x,y,z)}{U_3(y,z,w)}$、$\frac{U_1(y,z,w)}{U_2(y,z,w)}=\frac{H_2(y,w)G_1(y)}{H_1(z,w)G_2(z)}$、
$\frac{U_1(y,z,w)}{U_3(y,z,w)}=\frac{H_3(y,z)G_1(y)}{H_1(z,w)G_3(w)}$;

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U_1(y,z,w)H_1(z,w)G_2(z)G_3(w)dydzdw<\infty$

($\Rightarrow$)假設條件常態分佈族滿足相容性，因此根據(i)，

$$\begin{aligned}\frac{f_{1|234}(x|y,z,w)}{f_{2|134}(y|x,z,w)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_2(x,z,w)}\exp\{-\frac{(y-\mu_2(x,z,w))^2}{2\sigma_2^2(x,z,w)}\}}\\ &= \frac{\sigma_2(x,z,w)}{\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}+\frac{(y-\mu_2(x,z,w))^2}{2\sigma_2^2(x,z,w)}\}\\ &= \frac{\sigma_2(x,z,w)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\\ &\quad \exp\{\frac{y^2-2y\mu_2(x,z,w)}{2\sigma_2^2(x,z,w)}\}\exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}\} \qquad (11)\end{aligned}$$

可寫成(x,z,w)與(y,z,w)函數的相除，令

$$k_1(x,y,z,w)=\frac{-x^2+2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}+\frac{y^2-2y\mu_2(x,z,w)}{2\sigma_2^2(x,z,w)}$$

於是可假設存在$h_1(x,z,w)$與$h_2(y,z,w)$，使得$k_1(x,y,z,w)=h_1(x,z,w)+h_2(y,z,w)$

，由於$\frac{\partial}{\partial y}k_1(x,y,z,w)$是$(y,z,w)$的函數，而

$$\begin{aligned}\frac{\partial}{\partial y}k_1(x,y,z,w) &= \frac{-x^2}{2}\frac{\partial}{\partial y}[\frac{1}{\sigma_1^2(y,z,w)}]+x\frac{\partial}{\partial y}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}]+\frac{y}{\sigma_2^2(x,z,w)}\\ &\quad -\frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)} \qquad (12)\end{aligned}$$

因此，存在$C_1, \cdots, C_4$，使得

$$\frac{\partial}{\partial y}[\frac{1}{\sigma_1^2(y,z,w)}] = C_1(z,w)y + C_2(z,w) \text{且} \frac{\partial}{\partial y}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}] = C_3(z,w)y + C_4(z,w)$$

，積分可得

$$\frac{1}{\sigma_1^2(y,z,w)} = \frac{1}{2}C_1(z,w)y^2 + C_2(z,w)y + C_5(z,w) \tag{13}$$

$$\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)} = \frac{1}{2}C_3(z,w)y^2 + C_4(z,w)y + C_6(z,w) \tag{14}$$

將(13)、(14)式代回(12)式可得

$$\begin{aligned}(12) &= y[\frac{-1}{2}C_1(z,w)x^2 + C_3(z,w)x + \frac{1}{\sigma_2^2(x,z,w)}] - \frac{1}{2}C_2(z,w)x^2 + C_4(z,w)x \\ &- \frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)}\end{aligned}$$

所以，
$$\begin{aligned}-\frac{1}{2}C_1(z,w)x^2 + C_3(z,w)x + \frac{1}{\sigma_2^2(x,z,w)} &= C_7(z,w) \\ -\frac{1}{2}C_2(z,w)x^2 + C_4(z,w)x - \frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)} &= C_8(z,w)\end{aligned}$$

移項可得:

$$\frac{1}{\sigma_2^2(x,z,w)} = \frac{1}{2}C_1(z,w)x^2 - C_3(z,w)x + C_7(z,w) \tag{15}$$

$$\frac{\mu_2(x,z,w)}{\sigma_2^2(x,z,w)} = -\frac{1}{2}C_2(z,w)x^2 + C_4(z,w)x - C_8(z,w) \tag{16}$$

將(13)、(14)、(15)、(16)式代回(11)式:

$$\begin{aligned}(11) &= \frac{\sigma_2(x,z,w)}{\sigma_1(y,z,w)} \exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}\} \\ &\quad \exp\{-\frac{x^2 - 2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{y^2 - 2y\mu_2(x,z,w)}{2\sigma_2^2(x,z,w)}\} \\ &= \sigma_2(x,z,w) \exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}\} \frac{1}{\sigma_1(y,z,w)} \exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \\ &\quad \exp\{-\frac{1}{2}C_5(z,w)x^2 + C_6(z,w)x + \frac{1}{2}C_7(z,w)y^2 + C_8(z,w)y\} \\ &= \sigma_2(x,z,w) \exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}\} \exp\{-\frac{1}{2}C_5(z,w)x^2 + C_6(z,w)x\} \\ &\quad \frac{1}{\sigma_1(y,z,w)} \exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \exp\{\frac{1}{2}C_7(z,w)y^2 + C_8(z,w)y\}\end{aligned}$$

可令

$$V_1(x,z,w)=\sigma_2^2(x,z,w)\exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}-\frac{1}{2}C_5(z,w)x^2+C_6(z,w)x\} \tag{17}$$

$$[U_1(y,z,w)]^{-1}=\frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}+\frac{1}{2}C_7(z,w)y^2+C_8(z,w)y\} \tag{18}$$

$$\begin{aligned}\frac{f_{1|234}(x|y,z,w)}{f_{3|124}(z|x,y,w)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_3(x,y,w)}\exp\{-\frac{(z-\mu_3(x,y,w))^2}{2\sigma_3^2(x,y,w)}\}}\\ &= \frac{\sigma_3(x,y,w)}{\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}+\frac{(z-\mu_3(x,y,w))^2}{2\sigma_3^2(x,y,w)}\}\\ &= \frac{\sigma_3(x,y,w)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\\ &\quad \exp\{\frac{z^2-2z\mu_3(x,y,w)}{2\sigma_3^2(x,y,w)}\}\exp\{\frac{\mu_3^2(x,y,w)}{2\sigma_3^2(x,y,w)}\}\end{aligned} \tag{19}$$

可寫成(x,z,w)與(y,z,w)函數的相除，令

$$k_2(x,y,z,w)=\frac{-x^2+2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}+\frac{z^2-2z\mu_3(x,y,w)}{2\sigma_3^2(x,y,w)}$$

於是可假設存在$h_3(x,y,w)$與$h_2(y,z,w)$，使得$k_2(x,y,z,w)=h_3(x,y,w)+h_4(y,z,w)$

，由於$\frac{\partial}{\partial y}k_2(x,y,z,w)$是$(y,z,w)$的函數，而

$$\begin{aligned}\frac{\partial}{\partial z}k_2(x,y,z,w) &= \frac{-x^2}{2}\frac{\partial}{\partial z}[\frac{1}{\sigma_1^2(y,z,w)}]+x\frac{\partial}{\partial z}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}]+\frac{z}{\sigma_3^2(x,y,w)}\\ &\quad -\frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)}\end{aligned} \tag{20}$$

因此，存在$D_1,\cdots,D_4$，使得

$$\frac{\partial}{\partial z}[\frac{1}{\sigma_1^2(y,z,w)}]=D_1(y,w)y+D_2(y,w)\text{且}\frac{\partial}{\partial z}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}]=D_3(y,w)y+D_4(y,w)$$

，積分可得

$$
\frac{1}{\sigma_1^2(y,z,w)} = \frac{1}{2}D_1(y,w)z^2 + D_2(y,w)z + D_5(y,w) \tag{21}
$$

$$
\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)} = \frac{1}{2}D_3(y,w)z^2 + D_4(y,w)y + D_6(y,w) \tag{22}
$$

將(21)、(22)式帶回(20)式可得:

$$
\begin{aligned}
(20) &= z[\frac{-1}{2}D_1(y,w)x^2 + D_3(y,w)x + \frac{1}{\sigma_3^2(x,y,w)}] - \frac{1}{2}D_2(y,w)x^2 + D_4(y,w)x \\
&- \frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)}
\end{aligned}
$$

所以，
$$
\begin{aligned}
-\frac{1}{2}D_1(y,w)x^2 + D_3(y,w)x + \frac{1}{\sigma_3^2(x,y,w)} &= D_7(y,w) \\
-\frac{1}{2}D_2(y,w)x^2 + D_4(y,w)x - \frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)} &= D_8(y,w)
\end{aligned}
$$

移項可得:

$$
\frac{1}{\sigma_3^2(x,y,w)} = \frac{1}{2}D_1(y,w)x^2 - D_3(y,w)x + D_7(y,w) \tag{23}
$$

$$
\frac{\mu_3(x,y,w)}{\sigma_3^2(x,y,w)} = -\frac{1}{2}D_2(y,w)x^2 + D_4(y,w)x - D_8(y,w) \tag{24}
$$

將(21)、(22)、(23)、(24)式帶回(19)式:

$$
\begin{aligned}
(19) &= \frac{\sigma_3(x,y,w)}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{\mu_3^2(x,y,w)}{2\sigma_3^2(x,y,w)}\} \\
&\quad \exp\{-\frac{x^2 - 2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{z^2 - 2z\mu_3(x,y,w)}{2\sigma_3^2(x,y,w)}\} \\
&= \sigma_3(x,y,w)\exp\{\frac{\mu_3^2(x,y,w)}{2\sigma_3^2(x,y,w)}\}\frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \\
&\quad \exp\{-\frac{1}{2}D_5(y,w)x^2 + D_6(y,w)x + \frac{1}{2}D_7(y,w)z^2 + D_8(y,w)z\} \\
&= \sigma_3(x,y,w)\exp\{\frac{\mu_3^2(x,y,w)}{2\sigma_3^2(x,y,w)}\}\exp\{-\frac{1}{2}D_5(y,w)x^2 + D_6(y,w)x\} \\
&\quad \frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{1}{2}D_7(y,w)z^2 + D_8(y,w)z\}
\end{aligned} \tag{25}
$$

可令

$$V_2(x,y,w) = \sigma_3^2(x,y,w)\exp\{\frac{\mu_3^2(x,y,w)}{2\sigma_3^2(x,y,w)} - \frac{1}{2}D_5(y,w)x^2 + D_6(y,w)x\} \quad (26)$$

$$[U_2(y,z,w)]^{-1} = \frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{1}{2}D_7(y,w)z^2 + D_8(y,w)z\} \quad (27)$$

$$\begin{aligned}
\frac{f_{1|234}(x|y,z,w)}{f_{4|123}(w|x,y,z)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_4(x,y,z)}\exp\{-\frac{(w-\mu_4(x,y,z))^2}{2\sigma_4^2(x,y,z)}\}} \\
&= \frac{\sigma_4(x,y,z)}{\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)} + \frac{(w-\mu_4(x,y,z))^2}{2\sigma_4^2(x,y,z)}\} \\
&= \frac{\sigma_4(x,y,z)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \\
&\quad \exp\{\frac{w^2-2w\mu_4(x,y,z)}{2\sigma_4^2(x,y,z)}\}\exp\{\frac{\mu_4^2(x,y,z)}{2\sigma_4^2(x,y,z)}\} \qquad (28)
\end{aligned}$$

可寫成(x,y,z)與(y,z,w)函數的相除，令

$$k_3(x,y,z,w) = \frac{-x^2+2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{w^2-2w\mu_4(x,y,z)}{2\sigma_4^2(x,y,z)}$$

於是可假設存在$h_5(x,y,z)$與$h_6(y,z,w)$，使得$k_3(x,y,z,w) = h_5(x,y,z) + h_6(y,z,w)$

，由於$\frac{\partial}{\partial y}k_3(x,y,z,w)$是$(y,z,w)$的函數，而

$$\begin{aligned}
\frac{\partial}{\partial w}k_3(x,y,z,w) &= \frac{-x^2}{2}\frac{\partial}{\partial w}[\frac{1}{\sigma_1^2(y,z,w)}] + x\frac{\partial}{\partial w}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}] + \frac{w}{\sigma_4^2(x,y,z)} \\
&\quad -\frac{\mu_4(x,y,w)}{\sigma_4^2(x,y,z)} \qquad (29)
\end{aligned}$$

因此，存在$E_1,\cdots,E_4$，使得

$$\frac{\partial}{\partial w}[\frac{1}{\sigma_1^2(y,z,w)}] = E_1(y,z)w + E_2(y,z) \text{且} \frac{\partial}{\partial w}[\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)}] = E_3(y,z)w + E_4(y,z)$$

，積分可得

$$\frac{1}{\sigma_1^2(y,z,w)} = \frac{1}{2}E_1(y,z)w^2 + E_2(y,z)w + E_5(y,z) \quad (30)$$

$$\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)} = \frac{1}{2}E_3(y,z)w^2 + E_4(y,z)w + E_6(y,w) \quad (31)$$

將(30)、(31)式帶回(29)式可得:

$$
\begin{aligned}
(29) = w[\frac{-1}{2}E_1(y,z)x^2 + E_3(y,w)x + \frac{1}{\sigma_4^2(x,y,z)}] \\
-\frac{1}{2}E_2(y,z)x^2 + E_4(y,z)x - \frac{\mu_4(x,y,z)}{\sigma_4^2(x,y,z)}
\end{aligned}
$$

所以，
$$
\begin{aligned}
-\frac{1}{2}E_1(y,z)x^2 + E_3(y,z)x + \frac{1}{\sigma_4^2(x,y,z)} &= E_7(y,z) \\
-\frac{1}{2}E_2(y,z)x^2 + E_4(y,z)x - \frac{\mu_4(x,y,z)}{\sigma_4^2(x,y,z)} &= E_8(y,z)
\end{aligned}
$$

移項可得:

$$\frac{1}{\sigma_4^2(x,y,z)} = \frac{1}{2}E_1(y,z)x^2 - E_3(y,z)x + E_7(y,z) \qquad (32)$$

$$\frac{u_4(x,y,z)}{\sigma_4^2(x,y,z)} = -\frac{1}{2}E_2(y,z)x^2 + E_4(y,z)x - E_8(y,z) \qquad (33)$$

將(30)、(31)、(32)、(33)式帶回(28)式:

$$
\begin{aligned}
(28) &= \frac{\sigma_4(x,y,z)}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{\mu_4^2(x,y,z)}{2\sigma_4^2(x,y,z)}\} \\
&\quad \exp\{-\frac{x^2-2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{w^2-2w\mu_4(x,y,z)}{2\sigma_4^2(x,y,z)}\} \\
&= \sigma_4(x,y,z)\exp\{\frac{\mu_4^2(x,y,z)}{2\sigma_4^2(x,y,z)}\}\frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \\
&\quad \exp\{-\frac{1}{2}E_5(y,z)x^2 + E_6(y,z)x + \frac{1}{2}E_7(y,z)w^2 + E_8(y,z)w\} \\
&= \sigma_4(x,y,z)\exp\{\frac{\mu_4^2(x,y,z)}{2\sigma_4^2(x,y,z)}\}\exp\{-\frac{1}{2}E_5(y,z)x^2 + E_6(y,z)x\} \\
&\quad \frac{1}{\sigma_1(y,z,w)}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{1}{2}E_7(y,z)w^2 + E_8(y,z)w\}
\end{aligned}
\qquad (34)
$$

可令

$$V_3(x,y,z) = \sigma_4^2(x,y,z)\exp\{\frac{\mu_4^2(x,y,z)}{2\sigma_4^2(x,y,z)} - \frac{1}{2}E_5(y,z)x^2 + E_6(y,z)x\} \qquad (35)$$

$$[U_3(y,z,w)]^{-1} = \frac{1}{\sigma_1(y,z,w)} \exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{1}{2}E_7(y,z)w^2 + E_8(y,z)w\} \quad (36)$$

接著，可由(13)、(21)、(30)式得知

$$\begin{aligned}
\sigma_1^2(y,z,w) &= \frac{1}{2}C_1(z,w)y^2 + C_2(z,w)y + C_5(z,w) && (37)\\
&= \frac{1}{2}D_1(y,w)z^2 + D_2(y,w)z + D_5(y,w) && (38)\\
&= \frac{1}{2}E_1(y,z)w^2 + E_2(y,z)w + E_5(y,z) && (39)
\end{aligned}$$

觀察上述可知，y、z、w的最高次數皆爲二次，於是可存在常數使得C_{ij}、D_{ij}、E_{ij}

$$\begin{aligned}
(37) &= \frac{1}{2}[(C_{15}z^2 + C_{14}z + C_{13})(C_{12}w^2 + C_{11}w + C_{10})]y^2 + [(C_{25}z^2 + C_{24}z + C_{23})\\
&\quad (C_{22}w^2 + C_{21}w + C_{20})]y + [(C_{55}z^2 + C_{54}z + C_{53})(C_{52}w^2 + C_{51}w + C_{50})]\\
&= \frac{1}{2}C_{15}C_{12}y^2z^2w^2 + \frac{1}{2}C_{15}C_{11}y^2z^2w + \frac{1}{2}C_{15}C_{10}y^2z^2 + \frac{1}{2}C_{14}C_{12}y^2zw^2 +\\
&\quad \frac{1}{2}C_{14}C_{11}y^2zw + \frac{1}{2}C_{14}C_{10}y^2z + \frac{1}{2}C_{13}C_{12}y^2w^2 + \frac{1}{2}C_{13}C_{11}y^2w + \frac{1}{2}C_{13}C_{10}y^2\\
&\quad +C_{25}C_{22}yz^2w^2 + C_{25}C_{21}yz^2w + C_{25}C_{20}yz^2 + C_{24}C_{22}yzw^2 + C_{24}C_{21}yzw\\
&\quad +C_{24}C_{20}yz + C_{23}C_{22}yw^2 + C_{23}C_{21}yw + C_{23}C_{20}y + C_{55}C_{52}z^2w^2 + C_{55}C_{51}z^2w\\
&\quad +C_{55}C_{50}z^2 + C_{54}C_{52}zw^2 + C_{54}C_{51}zw + C_{54}C_{50}z + C_{53}C_{52}w^2 + C_{53}C_{51}w\\
&\quad +C_{53}C_{50}
\end{aligned}$$

$$\begin{aligned}
(38) &= \frac{1}{2}[(D_{15}y^2 + D_{14}y + D_{13})(D_{12}w^2 + D_{11}w + D_{10})]z^2 + [(D_{25}y^2 + D_{24}y + D_{23})\\
&\quad (D_{22}w^2 + D_{21}w + D_{20})]z + [(D_{55}y^2 + D_{54}y + D_{53})(D_{52}w^2 + D_{51}w + D_{50})]\\
&= \frac{1}{2}D_{15}C_{12}y^2z^2w^2 + \frac{1}{2}D_{15}D_{11}y^2z^2w + \frac{1}{2}D_{15}D_{10}y^2z^2 + \frac{1}{2}D_{14}D_{12}yz^2w^2 +\\
&\quad \frac{1}{2}D_{14}D_{11}yz^2w + \frac{1}{2}D_{14}D_{10}yz^2 + \frac{1}{2}D_{13}D_{12}z^2w^2 + \frac{1}{2}D_{13}D_{11}z^2w + \frac{1}{2}D_{13}D_{10}z^2\\
&\quad +D_{25}D_{22}y^2zw^2 + D_{25}D_{21}y^2zw + D_{25}D_{20}y^2z + D_{24}D_{22}yzw^2 + D_{24}D_{21}yzw +\\
&\quad D_{24}D_{20}yz + D_{23}D_{22}zw^2 + D_{23}D_{21}zw + D_{23}D_{20}z + D_{55}D_{52}y^2w^2 + D_{55}D_{51}y^2w\\
&\quad +D_{55}D_{50}y^2 + D_{54}D_{52}yw^2 + D_{54}D_{51}yw + D_{54}D_{50}y + D_{53}D_{52}w^2 + D_{53}D_{51}w\\
&\quad +D_{53}D_{50}
\end{aligned}$$

$$
\begin{aligned}
(39) \;=\; & \frac{1}{2}[(E_{15}y^2 + E_{14}y + E_{13})(E_{12}z^2 + E_{11}z + E_{10})]w^2 + [(E_{25}y^2 + E_{24}y + E_{23}) \\
& (E_{22}z^2 + E_{21}z + E_{20})]w + [(E_{55}y^2 + E_{54}y + E_{53})(E_{52}z^2 + E_{51}z + E_{50})] \\
= \; & \frac{1}{2}E_{15}E_{12}y^2z^2w^2 + \frac{1}{2}E_{15}E_{11}y^2zw^2 + \frac{1}{2}E_{15}E_{10}y^2w^2 + \frac{1}{2}E_{14}E_{12}yz^2w^2 + \\
& \frac{1}{2}E_{14}E_{11}yzw^2 + \frac{1}{2}E_{14}E_{10}yw^2 + \frac{1}{2}E_{13}E_{12}z^2w^2 + \frac{1}{2}E_{13}E_{11}zw^2 + \frac{1}{2}E_{13}E_{10}w^2 \\
& + E_{25}E_{22}y^2z^2w + E_{25}E_{21}y^2zw + E_{25}E_{20}y^2w + E_{24}E_{22}yz^2w + E_{24}E_{21}yzw + \\
& E_{24}E_{20}yw + E_{23}E_{22}z^2w + E_{23}E_{21}zw + E_{23}E_{20}w + E_{55}E_{52}y^2z^2 + E_{55}E_{51}y^2z \\
& + E_{55}E_{50}y^2 + E_{54}E_{52}yz^2 + E_{54}E_{51}yz + E_{54}E_{50}y + E_{53}E_{52}z^2 + E_{53}E_{51}z \\
& + E_{53}E_{50}
\end{aligned}
$$

比對上述三等式後可得知係數間有下列關係:

$C_{15}C_{12} = D_{15}D_{12} = E_{15}E_{12}$，$C_{15}C_{11} = D_{15}D_{11} = 2E_{25}E_{22}$，$C_{15}C_{10} = D_{15}D_{10} = 2E_{55}E_{52}$，$C_{14}C_{12} = 2D_{25}D_{22} = E_{15}E_{11}$，$C_{14}C_{11} = 2D_{25}D_{21} = 2E_{25}E_{21}$，$C_{14}C_{10} = 2D_{25}D_{20} = 2E_{55}E_{51}$，$C_{13}C_{12} = 2D_{55}D_{52} = E_{15}E_{10}$，$C_{13}C_{11} = 2D_{55}D_{51} = 2E_{25}E_{20}$，$C_{13}C_{10} = 2D_{55}D_{50} = 2E_{55}E_{50}$，$2C_{25}C_{22} = D_{14}D_{12} = E_{14}E_{12}$，$2C_{25}C_{21} = D_{14}D_{11} = 2E_{24}E_{22}$，$2C_{25}C_{20} = D_{14}D_{10} = 2E_{54}E_{52}$，$2C_{24}C_{22} = 2D_{24}D_{22} = E_{14}E_{11}$，$C_{24}C_{21} = D_{24}D_{21} = E_{24}E_{21}$，$C_{24}C_{20} = D_{24}D_{20} = E_{54}E_{51}$，$2C_{23}C_{22} = 2D_{54}D_{52} = E_{14}E_{10}$，$C_{23}C_{21} = D_{54}D_{51} = E_{24}E_{20}$，$C_{23}C_{20} = D_{54}D_{50} = E_{54}E_{50}$，$2C_{55}C_{52} = D_{13}D_{12} = E_{13}E_{12}$，$2C_{55}C_{51} = D_{13}D_{11} = 2E_{23}E_{22}$，$2C_{55}C_{50} = D_{13}D_{10} = 2E_{53}E_{52}$，$2C_{54}C_{52} = 2D_{23}D_{22} = E_{13}E_{11}$，$C_{54}C_{51} = D_{23}D_{21} = E_{23}E_{21}$，$C_{54}C_{50} = D_{23}D_{20} = E_{53}E_{51}$，$2C_{53}C_{52} = 2D_{53}D_{52} = E_{13}E_{10}$，$C_{53}C_{51} = D_{53}D_{51} = E_{23}E_{20}$，$C_{53}C_{50} = D_{53}D_{50} = E_{53}E_{50}$。

同樣的，可由(14)、(22)、(31)式得知

$$
\begin{aligned}
\frac{\mu_1(y,z,w)}{\sigma_1^2(y,z,w)} &= \frac{1}{2}C_3(z,w)y^2 + C_4(z,w)y + C_6(z,w) && (40) \\
&= \frac{1}{2}D_3(y,w)z^2 + D_4(y,w)z + D_6(y,w) && (41) \\
&= \frac{1}{2}E_3(y,z)w^2 + E_4(y,z)w + E_6(y,z) && (42)
\end{aligned}
$$

$$
\begin{aligned}
(40) \quad = \quad & \frac{1}{2}[(C_{35}z^2+C_{34}z+C_{33})(C_{32}w^2+C_{31}w+C_{30})]y^2+[(C_{45}z^2+C_{44}z+C_{43}) \\
& (C_{42}w^2+C_{41}w+C_{40})]y+[(C_{65}z^2+C_{64}z+C_{63})(C_{62}w^2+C_{61}w+C_{60})] \\
= \quad & \frac{1}{2}C_{35}C_{32}y^2z^2w^2+\frac{1}{2}C_{35}C_{31}y^2z^2w+\frac{1}{2}C_{35}C_{30}y^2z^2+\frac{1}{2}C_{34}C_{32}y^2zw^2+ \\
& \frac{1}{2}C_{34}C_{31}y^2zw+\frac{1}{2}C_{34}C_{30}y^2z+\frac{1}{2}C_{33}C_{32}y^2w^2+\frac{1}{2}C_{33}C_{31}y^2w+\frac{1}{2}C_{33}C_{30}y^2 \\
& +C_{45}C_{42}yz^2w^2+C_{45}C_{41}yz^2w+C_{45}C_{40}yz^2+C_{44}C_{42}yzw^2+C_{44}C_{41}yzw+ \\
& C_{44}C_{40}yz+C_{43}C_{42}yw^2+C_{43}C_{41}yw+C_{43}C_{40}y+C_{65}C_{62}z^2w^2+C_{65}C_{61}z^2w \\
& +C_{65}C_{60}z^2+C_{64}C_{62}zw^2+C_{64}C_{61}zw+C_{64}C_{60}z+C_{63}C_{62}w^2+C_{63}C_{61}w \\
& +C_{63}C_{60}
\end{aligned}
$$

$$
\begin{aligned}
(41) \quad = \quad & \frac{1}{2}[(D_{35}y^2+D_{34}y+D_{33})(D_{32}w^2+D_{31}w+D_{30})]z^2+[(D_{45}y^2+D_{44}y+D_{43}) \\
& (D_{42}w^2+D_{41}w+D_{40})]z+[(D_{65}y^2+D_{64}y+D_{63})(D_{62}w^2+D_{61}w+D_{60})] \\
= \quad & \frac{1}{2}D_{35}C_{32}y^2z^2w^2+\frac{1}{2}D_{35}D_{31}y^2z^2w+\frac{1}{2}D_{35}D_{30}y^2z^2+\frac{1}{2}D_{34}D_{32}yz^2w^2+ \\
& \frac{1}{2}D_{34}D_{31}yz^2w+\frac{1}{2}D_{34}D_{30}yz^2+\frac{1}{2}D_{33}D_{32}z^2w^2+\frac{1}{2}D_{33}D_{31}z^2w+\frac{1}{2}D_{33}D_{30}z^2 \\
& +D_{45}D_{42}y^2zw^2+D_{45}D_{41}y^2zw+D_{45}D_{40}y^2z+D_{44}D_{42}yzw^2+D_{44}D_{41}yzw \\
& +D_{44}D_{40}yz+D_{43}D_{42}zw^2+D_{43}D_{41}zw+D_{43}D_{40}z+D_{65}D_{62}y^2w^2+ \\
& D_{65}D_{61}y^2w+D_{65}D_{60}y^2+D_{64}D_{62}yw^2+D_{64}D_{61}yw+D_{64}D_{60}y+D_{63}D_{62}w^2 \\
& +D_{63}D_{61}w+D_{63}D_{60}
\end{aligned}
$$

$$
\begin{aligned}
(42) \quad = \quad & \frac{1}{2}[(E_{35}y^2+E_{34}y+E_{33})(E_{32}z^2+E_{31}z+E_{30})]w^2+[(E_{45}y^2+E_{44}y+E_{43}) \\
& (E_{42}z^2+E_{41}z+E_{40})]w+[(E_{65}y^2+E_{64}y+E_{63})(E_{62}z^2+E_{61}z+E_{60})] \\
= \quad & \frac{1}{2}E_{35}E_{32}y^2z^2w^2+\frac{1}{2}E_{35}E_{31}y^2zw^2+\frac{1}{2}E_{35}E_{30}y^2w^2+\frac{1}{2}E_{34}E_{32}yz^2w^2+ \\
& \frac{1}{2}E_{34}E_{31}yzw^2+\frac{1}{2}E_{34}E_{30}yw^2+\frac{1}{2}E_{33}E_{32}z^2w^2+\frac{1}{2}E_{33}E_{31}zw^2+\frac{1}{2}E_{33}E_{30}w^2 \\
& +E_{45}E_{42}y^2z^2w+E_{45}E_{41}y^2zw+E_{45}E_{40}y^2w+E_{44}E_{42}yz^2w+E_{44}E_{41}yzw \\
& +E_{44}E_{40}yw+E_{43}E_{42}z^2w+E_{43}E_{41}zw+E_{43}E_{40}w+E_{65}E_{62}y^2z^2+ \\
& E_{65}E_{61}y^2z+E_{65}E_{60}y^2+E_{64}E_{62}yz^2+E_{64}E_{61}yz+E_{64}E_{60}y+E_{63}E_{62}z^2 \\
& +E_{63}E_{61}z+E_{63}E_{60}
\end{aligned}
$$

比對上述三等式後可得知係數間有下列關係:

$C_{35}C_{32}=D_{35}D_{32}=E_{35}E_{32}$，$C_{35}C_{31}=D_{35}D_{31}=2E_{45}E_{42}$，$C_{35}C_{30}=D_{35}D_{30}=$

$2E_{65}E_{62}$，$C_{34}C_{32} = 2D_{45}D_{42} = E_{35}E_{31}$，$C_{34}C_{31} = 2D_{45}D_{41} = 2E_{45}E_{41}$，$C_{34}C_{30} = 2D_{45}D_{40} = 2E_{65}E_{61}$，$C_{33}C_{32} = 2D_{65}D_{62} = E_{35}E_{30}$，$C_{33}C_{31} = 2D_{65}D_{61} = 2E_{45}E_{40}$，$C_{33}C_{30} = 2D_{65}D_{60} = 2E_{65}E_{60}$，$2C_{45}C_{42} = D_{34}D_{32} = E_{34}E_{32}$，$2C_{45}C_{41} = D_{34}D_{31} = 2E_{44}E_{42}$，$2C_{45}C_{40} = D_{34}D_{30} = 2E_{64}E_{62}$，$2C_{44}C_{42} = 2D_{44}D_{42} = E_{34}E_{31}$，$C_{44}C_{41} = D_{44}D_{41} = E_{44}E_{41}$，$C_{44}C_{40} = D_{44}D_{40} = E_{64}E_{61}$，$2C_{43}C_{42} = 2D_{64}D_{62} = E_{34}E_{30}$，$C_{43}C_{41} = D_{64}D_{61} = E_{44}E_{40}$，$C_{43}C_{40} = D_{64}D_{60} = E_{64}E_{60}$，$2C_{65}C_{62} = D_{33}D_{32} = E_{33}E_{32}$，$2C_{65}C_{61} = D_{33}D_{31} = 2E_{43}E_{42}$，$2C_{65}C_{60} = D_{33}D_{30} = 2E_{63}E_{62}$，$2C_{64}C_{62} = 2D_{43}D_{42} = E_{33}E_{31}$，$C_{64}C_{61} = D_{43}D_{41} = E_{43}E_{41}$，$C_{64}C_{60} = D_{43}D_{40} = E_{63}E_{61}$，$2C_{63}C_{62} = 2D_{63}D_{62} = E_{33}E_{30}$，$C_{63}C_{61} = D_{63}D_{61} = E_{43}E_{40}$，$C_{63}C_{60} = D_{63}D_{60} = E_{63}E_{60}$。

由(i)可知，$\dfrac{U_1(y,z,w)}{U_2(y,z,w)} = \dfrac{H_2(y,w)G_1(y)}{H_1(z,w)G_2(z)}$，所以將(18)、(27)式代入:

$$\frac{U_1(y,z,w)}{U_2(y,z,w)} = \frac{\sigma_1(y,z,w)\exp\{-\frac{1}{2}C_7(z,w)y^2 + C_8(z,w)y + \frac{1}{2}[\frac{u_1^2(y,z,w)}{\sigma_1^2(y,z,w)}]\}}{\sigma_1(y,z,w)\exp\{-\frac{1}{2}D_7(y,w)z^2 + D_8(y,w)z + \frac{1}{2}[\frac{u_1^2(y,z,w)}{\sigma_1^2(y,z,w)}]\}}$$

$$= \frac{\begin{array}{l}\exp\{-\frac{1}{2}C_{75}C_{72}y^2z^2w^2 - \frac{1}{2}C_{75}C_{71}y^2z^2w - \frac{1}{2}C_{75}C_{70}y^2z^2 - \frac{1}{2}C_{74}C_{72}y^2zw^2 \\ -\frac{1}{2}C_{74}C_{71}y^2zw - \frac{1}{2}C_{74}C_{70}y^2z - \frac{1}{2}C_{73}C_{72}y^2w^2 - \frac{1}{2}C_{73}C_{71}y^2w - \frac{1}{2}C_{73}C_{70}y^2 \\ -C_{85}C_{82}y^2z^2w^2 - C_{85}C_{81}yz^2w - C_{85}C_{80}yz^2 - C_{84}C_{82}yzw^2 - C_{84}C_{81}yzw \\ -C_{84}C_{80}yz - C_{83}C_{82}yw^2 - C_{83}C_{81}yw - C_{83}C_{80}y\}\end{array}}{\begin{array}{l}\exp\{-\frac{1}{2}D_{75}D_{72}y^2z^2w^2 - \frac{1}{2}D_{75}D_{71}y^2z^2w - \frac{1}{2}D_{75}D_{70}y^2z^2 - \frac{1}{2}D_{74}D_{72}yz^2w^2 \\ -\frac{1}{2}D_{74}D_{71}yz^2w - \frac{1}{2}D_{74}D_{70}yz^2 - \frac{1}{2}D_{73}D_{72}z^2w^2 - \frac{1}{2}D_{73}D_{71}z^2w - \frac{1}{2}D_{73}D_{70}z^2 \\ -D_{85}D_{82}y^2zw^2 - D_{85}D_{81}y^2zw - D_{85}D_{80}y^2z - D_{84}D_{82}yzw^2 - D_{84}D_{81}yzw \\ -D_{84}D_{80}yz - D_{83}D_{82}zw^2 - D_{83}D_{81}zw - D_{83}D_{80}z\}\end{array}}$$

於是，可得到:

$C_{75}C_{72} = D_{75}D_{72}$，$C_{75}C_{71} = D_{75}D_{71}$，$C_{75}C_{70} = D_{75}D_{70}$，$C_{74}C_{72} = 2D_{85}D_{82}$，$C_{74}C_{71} = 2D_{85}D_{81}$，$C_{74}C_{70} = 2D_{85}D_{80}$，$2C_{85}C_{82} = D_{74}D_{72}$，$2C_{85}C_{81} = D_{74}D_{71}$，$2C_{85}C_{80} = D_{74}D_{70}$，$C_{84}C_{82} = D_{84}D_{82}$，$C_{84}C_{81} = D_{84}D_{81}$，$C_{84}C_{80} = D_{84}D_{80}$。

同樣的，$\dfrac{U_1(y,z,w)}{U_3(y,z,w)} = \dfrac{H_3(y,z)G_1(y)}{H_1(z,w)G_3(w)}$，將(18)、(36)式代入可得:

$$\frac{U_1(y,z,w)}{U_3(y,z,w)} = \frac{\sigma_1(y,z,w)\exp\{-\frac{1}{2}C_7(z,w)y^2 + C_8(z,w)y + \frac{1}{2}[\frac{u_1^2(y,z,w)}{\sigma_1^2(y,z,w)}]\}}{\sigma_1(y,z,w)\exp\{-\frac{1}{2}E_7(y,z)w^2 + E_8(y,z)w + \frac{1}{2}[\frac{u_1^2(y,z,w)}{\sigma_1^2(y,z,w)}]\}}$$

$$= \frac{\begin{array}{l}\exp\{-\frac{1}{2}C_{75}C_{72}y^2z^2w^2 - \frac{1}{2}C_{75}C_{71}y^2z^2w - \frac{1}{2}C_{75}C_{70}y^2z^2 - \frac{1}{2}C_{74}C_{72}y^2zw^2 \\ -\frac{1}{2}C_{74}C_{71}y^2zw - \frac{1}{2}C_{74}C_{70}y^2z - \frac{1}{2}C_{73}C_{72}y^2w^2 - \frac{1}{2}C_{73}C_{71}y^2w - \frac{1}{2}C_{73}C_{70}y^2 \\ -C_{85}C_{82}y^2z^2w^2 - C_{85}C_{81}yz^2w - C_{85}C_{80}yz^2 - C_{84}C_{82}yzw^2 - C_{84}C_{81}yzw \\ -C_{84}C_{80}yz - C_{83}C_{82}yw^2 - C_{83}C_{81}yw - C_{83}C_{80}y\}\end{array}}{\begin{array}{l}\exp\{-\frac{1}{2}E_{75}E_{72}y^2z^2w^2 - \frac{1}{2}E_{75}E_{71}y^2zw^2 - \frac{1}{2}E_{75}E_{70}y^2w^2 - \frac{1}{2}E_{74}E_{72}yz^2w^2 \\ -\frac{1}{2}E_{74}E_{71}yzw^2 - \frac{1}{2}E_{74}E_{70}yw^2 - \frac{1}{2}E_{73}E_{72}z^2w^2 - \frac{1}{2}E_{73}E_{71}zw^2 - \frac{1}{2}E_{73}E_{70}w^2 \\ -E_{85}E_{82}y^2z^2w - E_{85}E_{81}y^2zw - E_{85}E_{80}y^2w - E_{84}E_{82}yz^2w - E_{84}E_{81}yzw \\ -E_{84}E_{80}yw - E_{83}E_{82}z^2w - E_{83}E_{81}zw - E_{83}E_{80}w\}\end{array}}$$

所以，可得到:

$C_{75}C_{72} = E_{75}E_{72}$，$C_{75}C_{71} = 2E_{85}E_{82}$，$C_{74}C_{72} = E_{75}E_{71}$，$C_{74}C_{71} = 2E_{85}E_{81}$，$C_{73}C_{72} = E_{75}E_{70}$，$C_{73}C_{71} = 2E_{85}E_{80}$，$2C_{85}C_{82} = E_{74}E_{72}$，$C_{85}C_{81} = E_{84}E_{82}$，$2C_{84}C_{82} = E_{74}E_{71}$，$C_{84}C_{81} = E_{84}E_{81}$，$2C_{83}C_{82} = E_{74}E_{70}$，$C_{83}C_{81} = E_{84}E_{80}$。

接著，$H_1(z,w)$必須相同，因此可知，$D_{73}D_{72} = E_{73}E_{72}$，$D_{73}D_{72} = 2E_{83}E_{82}$，$2D_{83}D_{82} = E_{73}E_{71}$，$D_{83}D_{81} = E_{83}E_{81}$，所以得到:

$$H_1(z,w) = \exp\{\frac{1}{2}D_{73}D_{72}z^2w^2 - \frac{1}{2}D_{73}D_{71}z^2w - D_{83}D_{82}zw^2 - D_{83}D_{81}zw\} \quad (43)$$

$$G_2(z) = \exp\{-\frac{1}{2}D_{73}D_{70}z^2 - D_{83}D_{80}z\} \quad (44)$$

$$G_3(w) = \exp\{-\frac{1}{2}E_{73}E_{70}w^2 - E_{83}E_{80}w\} \quad (45)$$

將上述得到的結果整理後，可得到以下結果:

$$\begin{aligned}&\sigma_1^2(y,z,w) = \{\frac{1}{2}C_1(z,w)y^2 + C_2(z,w)y + C_5(z,w)\}^{-1} \\ = \; &\{\frac{1}{2}[(C_{15}z^2 + C_{14}z + C_{13})(C_{12}w^2 + C_{11}w + C_{10})]y^2 + [(C_{25}z^2 + C_{24}z + C_{23}) \\ &(C_{22}w^2 + C_{21}w + C_{20})]y + [(C_{55}z^2 + C_{54}z + C_{53})(C_{52}w^2 + C_{51}w + C_{50})]\}^{-1}\end{aligned}$$

$$= \{\frac{1}{2}C_{15}C_{12}y^2z^2w^2 + \frac{1}{2}C_{15}C_{11}y^2z^2w + \frac{1}{2}C_{15}C_{10}y^2z^2 + \frac{1}{2}C_{14}C_{12}y^2zw^2 + \frac{1}{2}C_{14}C_{11}y^2zw + \frac{1}{2}C_{14}C_{10}y^2z + \frac{1}{2}C_{13}C_{12}y^2w^2 + \frac{1}{2}C_{13}C_{11}y^2w + \frac{1}{2}C_{13}C_{10}y^2 + C_{25}C_{22}yz^2w^2 + C_{25}C_{21}yz^2w + C_{25}C_{20}yz^2 + C_{24}C_{22}yzw^2 + C_{24}C_{21}yzw + C_{24}C_{20}yz + C_{23}C_{22}yw^2 + C_{23}C_{21}yw + C_{23}C_{20}y + C_{55}C_{52}z^2w^2 + C_{55}C_{51}z^2w + C_{55}C_{50}z^2 + C_{54}C_{52}zw^2 + C_{54}C_{51}zw + C_{54}C_{50}z + C_{53}C_{52}w^2 + C_{53}C_{51}w + C_{53}C_{50}\}^{-1}$$

$$\sigma_2^2(x,z,w) = \{\frac{1}{2}C_1(z,w)x^2 - C_3(z,w)x + C_7(z,w)\}^{-1}$$

$$= \{\frac{1}{2}C_{15}C_{12}x^2z^2w^2 + \frac{1}{2}C_{15}C_{11}x^2z^2w + \frac{1}{2}C_{15}C_{10}x^2z^2 + \frac{1}{2}C_{14}C_{12}x^2zw^2 + \frac{1}{2}C_{14}C_{11}x^2zw + \frac{1}{2}C_{14}C_{10}x^2z + \frac{1}{2}C_{13}C_{12}x^2w^2 + \frac{1}{2}C_{13}C_{11}x^2w + \frac{1}{2}C_{13}C_{10}x^2 - C_{35}C_{32}xz^2w^2 - C_{35}C_{31}xz^2w - C_{35}C_{30}xz^2 - C_{34}C_{32}xzw^2 - C_{34}C_{31}xzw - C_{34}C_{30}xz - C_{33}C_{32}xw^2 - C_{33}C_{31}xw - C_{33}C_{30}x + C_{75}C_{72}z^2w^2 + C_{75}C_{71}z^2w + C_{75}C_{70}z^2 + C_{74}C_{72}zw^2 + C_{74}C_{71}zw + C_{74}C_{70}z + C_{73}C_{72}w^2 + C_{73}C_{71}w + C_{73}C_{70}\}^{-1}$$

$$\sigma_3^2(x,y,w) = \{\frac{1}{2}D_1(y,w)x^2 - D_3(y,w)x + D_7(y,w)\}^{-1}$$

$$= \{\frac{1}{2}D_{15}D_{12}x^2y^2w^2 + \frac{1}{2}D_{15}D_{11}x^2y^2w + \frac{1}{2}D_{15}D_{10}x^2y^2 + \frac{1}{2}D_{14}D_{12}x^2yw^2 + \frac{1}{2}D_{14}D_{11}x^2yw + \frac{1}{2}D_{14}D_{10}x^2y + \frac{1}{2}D_{13}D_{12}x^2w^2 + \frac{1}{2}D_{13}D_{11}x^2w + \frac{1}{2}D_{13}D_{10}x^2 - D_{35}D_{32}xy^2w^2 - D_{35}D_{31}xy^2w - D_{35}D_{30}xy^2 - D_{34}D_{32}xyw^2 - D_{34}D_{31}xyw - D_{34}D_{30}xy - D_{33}D_{32}xw^2 - D_{33}D_{31}xw - D_{33}D_{30}x + D_{75}D_{72}y^2w^2 + D_{75}D_{71}y^2w + D_{75}D_{70}y^2 + D_{74}D_{72}yw^2 + D_{74}D_{71}yw + D_{74}D_{70}y + D_{73}D_{72}w^2 + D_{73}D_{71}w + D_{73}D_{70}\}^{-1}$$

$$= \{\frac{1}{2}C_{15}C_{12}x^2y^2w^2 + \frac{1}{2}C_{15}C_{11}x^2y^2w + \frac{1}{2}C_{15}C_{10}x^2y^2 + C_{25}C_{22}x^2yw^2 + C_{25}C_{21}x^2yw + C_{25}C_{20}x^2y + C_{55}C_{52}x^2w^2 + C_{55}C_{51}x^2w + C_{55}C_{50}x^2 - C_{35}C_{32}xy^2w^2 - C_{35}C_{31}xy^2w - C_{35}C_{30}xy^2 - 2C_{45}C_{42}xyw^2 - 2C_{45}C_{41}xyw - 2C_{45}C_{40}xy2C_{65}C_{62}xw^2 - 2C_{65}C_{61}xw - 2C_{65}C_{60}x + C_{75}C_{72}y^2w^2 + C_{75}C_{71}y^2w$$

$$+C_{75}C_{70}y^2 + 2C_{85}C_{82}yw^2 + 2C_{85}C_{81}yw + 2C_{85}C_{80}y + D_{73}D_{72}w^2 + D_{73}D_{71}w$$
$$+D_{73}D_{70}\}^{-1}$$

$$\sigma_4^2(x,y,z) = \{\frac{1}{2}E_1(y,z)x^2 - E_3(y,z)x + E_7(y,z)\}^{-1}$$

$$= \{\frac{1}{2}E_{15}E_{12}x^2y^2z^2 + \frac{1}{2}E_{15}E_{11}x^2y^2z + \frac{1}{2}E_{15}E_{10}x^2y^2 + \frac{1}{2}E_{14}E_{12}x^2yz^2 +$$
$$\frac{1}{2}E_{14}E_{11}x^2yz + \frac{1}{2}E_{14}E_{10}x^2y + \frac{1}{2}E_{13}E_{12}x^2z^2 + \frac{1}{2}E_{13}E_{11}x^2z + \frac{1}{2}E_{13}E_{10}x^2 -$$
$$E_{35}E_{32}xy^2z^2 - E_{35}E_{31}xy^2z - E_{35}E_{30}xy^2 - E_{34}E_{32}xyz^2 - E_{34}E_{31}xyw -$$
$$E_{34}E_{30}xy - E_{33}E_{32}xz^2 - E_{33}E_{31}xz - E_{33}E_{30}x + E_{75}E_{72}y^2z^2 + E_{75}E_{71}y^2z$$
$$+E_{75}E_{70}y^2 + E_{74}E_{72}yz^2 + E_{74}E_{71}yz + E_{74}E_{70}y + E_{73}E_{72}z^2 + E_{73}E_{71}z$$
$$+E_{73}E_{70}\}^{-1}$$

$$= \{\frac{1}{2}C_{15}C_{12}x^2y^2z^2 + \frac{1}{2}C_{14}C_{12}x^2y^2z + \frac{1}{2}C_{13}C_{12}x^2y^2 + C_{25}C_{22}x^2yz^2 +$$
$$C_{24}C_{22}x^2yz + C_{23}C_{22}x^2y + C_{55}C_{52}x^2z^2 + C_{54}C_{52}x^2z + C_{53}C_{52}x^2 -$$
$$C_{35}C_{32}xy^2z^2 - C_{34}C_{32}xy^2z - C_{33}C_{32}xy^2 - 2C_{45}C_{42}xyz^2 - 2C_{44}C_{42}xyw -$$
$$2C_{43}C_{42}xy - 2C_{65}C_{62}xz^2 - 2C_{64}C_{62}xz - 2C_{63}C_{62}x + C_{75}C_{72}y^2z^2 + C_{74}C_{72}y^2z$$
$$+C_{73}C_{72}y^2 + 2C_{85}C_{82}yz^2 + 2C_{84}C_{82}yz + 2C_{83}C_{82}y + D_{73}D_{72}z^2 + 2D_{83}D_{82}z$$
$$+E_{73}E_{70}\}^{-1}$$

$$\frac{\mu_1^2(y,z,w)}{\sigma_1^2(y,z,w)} = \frac{1}{2}C_3(z,w)y^2 + C_4(z,w)y + C_6(z,w)$$

$$= \frac{1}{2}C_{35}C_{32}y^2z^2w^2 + \frac{1}{2}C_{35}C_{31}y^2z^2w + \frac{1}{2}C_{35}C_{30}y^2z^2 + \frac{1}{2}C_{34}C_{32}y^2zw^2 +$$
$$\frac{1}{2}C_{34}C_{31}y^2zw + \frac{1}{2}C_{34}C_{30}y^2z + \frac{1}{2}C_{33}C_{32}y^2w^2 + \frac{1}{2}C_{33}C_{31}y^2w + \frac{1}{2}C_{33}C_{30}y^2$$
$$+C_{45}C_{42}yz^2w^2 + C_{45}C_{41}yz^2w + C_{45}C_{40}yz^2 + C_{44}C_{42}yzw^2 + C_{44}C_{41}yzw +$$
$$C_{44}C_{40}yz + C_{43}C_{42}yw^2 + C_{43}C_{41}yw + C_{43}C_{40}y + C_{65}C_{62}z^2w^2 + C_{65}C_{61}z^2w$$
$$+C_{65}C_{60}z^2 + C_{64}C_{62}zw^2 + C_{64}C_{61}zw + C_{64}C_{60}z + C_{63}C_{62}w^2 + C_{63}C_{61}w$$
$$+C_{63}C_{60}$$

$$\frac{\mu_2^2(x,z,w)}{\sigma_2^2(x,z,w)} = -\frac{1}{2}C_2(z,w)x^2 + C_4(z,w)y - C_8(z,w)$$

$$= -\frac{1}{2}C_{25}C_{22}x^2z^2w^2 - \frac{1}{2}C_{25}C_{21}x^2z^2w - \frac{1}{2}C_{25}C_{20}x^2z^2 - \frac{1}{2}C_{24}C_{22}x^2zw^2 -$$

$$\frac{1}{2}C_{24}C_{21}x^2zw + -\frac{1}{2}C_{24}C_{20}x^2z - \frac{1}{2}C_{23}C_{22}x^2w^2 - \frac{1}{2}C_{23}C_{21}x^2w - \frac{1}{2}C_{23}C_{20}x^2$$

$$+C_{45}C_{42}xz^2w^2 + C_{45}C_{41}xz^2w + C_{45}C_{40}xz^2 + C_{44}C_{42}xzw^2 + C_{44}C_{41}xzw$$

$$+C_{44}C_{40}xz + C_{43}C_{42}xw^2 + C_{43}C_{41}xw + C_{43}C_{40}x - C_{85}C_{82}z^2w^2 - C_{85}C_{81}z^2w$$

$$-C_{85}C_{80}z^2 - C_{84}C_{82}zw^2 - C_{84}C_{81}zw - C_{84}C_{80}z - C_{83}C_{82}w^2 - C_{83}C_{81}w$$

$$-C_{83}C_{80}$$

$$\frac{\mu_3^2(x,y,w)}{\sigma_3^2(x,y,w)} = -\frac{1}{2}D_2(y,w)x^2 + D_4(y,w)x - D_8(y,w)$$

$$= -\frac{1}{2}D_{25}D_{22}x^2y^2w^2 - \frac{1}{2}D_{25}D_{21}x^2y^2w - \frac{1}{2}D_{25}D_{20}x^2y^2 - \frac{1}{2}D_{24}D_{22}x^2yw^2 -$$

$$\frac{1}{2}D_{24}D_{21}x^2yw - \frac{1}{2}D_{24}D_{20}x^2y - \frac{1}{2}D_{23}D_{22}x^2w^2 - \frac{1}{2}D_{23}D_{21}x^2w - \frac{1}{2}D_{23}D_{20}x^2$$

$$+D_{45}D_{42}xy^2w^2 + D_{45}D_{41}xy^2w + D_{45}D_{40}xy^2 + D_{44}D_{42}xyw^2 + D_{44}D_{41}xyw$$

$$+D_{44}D_{40}xy + D_{43}D_{42}xw^2 + D_{43}D_{41}xw + D_{43}D_{40}x - D_{85}D_{82}y^2w^2 -$$

$$D_{85}D_{81}y^2w - D_{85}D_{80}y^2 - D_{84}D_{82}yw^2 - D_{84}D_{81}yw - D_{84}D_{80}y - D_{83}D_{82}w^2$$

$$-D_{83}D_{81}w - D_{83}D_{80}$$

$$= -\frac{1}{4}C_{14}C_{12}x^2y^2w^2 - \frac{1}{4}C_{14}C_{11}x^2y^2w - \frac{1}{4}C_{14}C_{10}x^2y^2 - \frac{1}{2}C_{24}C_{22}x^2yw^2 -$$

$$\frac{1}{2}C_{24}C_{21}x^2yw - \frac{1}{2}C_{24}C_{20}x^2y - \frac{1}{2}C_{54}C_{52}x^2w^2 - \frac{1}{2}C_{54}C_{51}x^2w - \frac{1}{2}C_{54}C_{50}x^2 +$$

$$\frac{1}{2}C_{34}C_{32}xy^2w^2 + \frac{1}{2}C_{34}C_{31}xy^2w + \frac{1}{2}C_{34}C_{30}xy^2 + C_{44}C_{42}xyw^2 +$$

$$C_{44}C_{41}xyw + C_{44}C_{40}xy + C_{64}C_{62}xw^2 + C_{64}C_{61}xw + C_{64}C_{60}x - \frac{1}{2}C_{74}C_{72}y^2w^2$$

$$-\frac{1}{2}C_{74}C_{71}y^2w - \frac{1}{2}C_{74}C_{70}y^2 - C_{84}C_{82}yw^2 - C_{84}C_{81}yw - C_{84}C_{80}y - D_{83}D_{82}w^2$$

$$-D_{83}D_{81}w - D_{83}D_{80}$$

$$\frac{\mu_4^2(x,y,z)}{\sigma_4^2(x,y,z)} = -\frac{1}{2}E_2(y,z)x^2 + E_4(y,z)x - E_8(y,z)$$

$$= -\frac{1}{2}E_{25}E_{22}x^2y^2z^2 - \frac{1}{2}E_{25}E_{21}x^2y^2z - \frac{1}{2}E_{25}E_{20}x^2y^2 - \frac{1}{2}E_{24}E_{22}x^2yz^2 -$$

$$\frac{1}{2}E_{24}E_{21}x^2yz - \frac{1}{2}E_{24}E_{20}x^2y - \frac{1}{2}E_{23}E_{22}x^2z^2 - \frac{1}{2}E_{23}E_{21}x^2z - \frac{1}{2}E_{23}E_{20}x^2+$$

$$
\begin{aligned}
& E_{45}E_{42}xy^2z^2 + E_{45}E_{41}xy^2z + E_{45}E_{40}xy^2 + E_{44}E_{42}xyz^2 + E_{44}E_{41}xyw + \\
& E_{44}E_{40}xy + E_{43}E_{42}xz^2 + E_{43}E_{41}xz + E_{43}E_{40}x - E_{85}E_{82}y^2z^2 - E_{85}E_{81}y^2z - \\
& E_{85}E_{80}y^2 - E_{84}E_{82}yz^2 - E_{84}E_{81}yz - E_{84}E_{80}y - E_{83}E_{82}z^2 - E_{83}E_{81}z - \\
& E_{83}E_{80} \\
= & -\frac{1}{4}C_{15}C_{11}x^2y^2z^2 - \frac{1}{4}C_{14}C_{11}x^2y^2z - \frac{1}{4}C_{13}C_{11}x^2y^2 - \frac{1}{2}C_{25}C_{21}x^2yz^2 - \\
& \frac{1}{2}C_{24}C_{21}x^2yz - \frac{1}{2}C_{23}C_{21}x^2y - \frac{1}{2}C_{55}C_{51}x^2z^2 - \frac{1}{2}C_{54}C_{51}x^2z - \frac{1}{2}C_{53}C_{51}x^2 + \\
& \frac{1}{2}C_{35}C_{31}xy^2z^2 + \frac{1}{2}C_{34}C_{31}xy^2z + \frac{1}{2}C_{33}C_{31}xy^2 + C_{45}C_{41}xyz^2 + C_{44}C_{41}xyw \\
& +C_{43}C_{41}xy + C_{65}C_{61}xz^2 + C_{64}C_{61}xz + C_{63}C_{61}x - \frac{1}{2}C_{75}C_{71}y^2z^2 - \frac{1}{2}C_{74}C_{71}y^2z \\
& -\frac{1}{2}C_{73}C_{71}y^2 - C_{85}C_{81}yz^2 - C_{84}C_{81}yz - C_{83}C_{81}y - \frac{1}{2}D_{73}D_{71}z^2 - D_{83}D_{81}z - \\
& E_{83}E_{80}
\end{aligned}
$$

我們可造出聯合密度函數

$$
\begin{aligned}
& f(x,y,z,w) \propto f_{1|234}(x|y,z,w)U_1(y,z,w)H_1(z,w)G_2(z)G_3(w) \\
= & \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{4}C_{15}C_{12}x^2y^2z^2w^2 - \frac{1}{4}C_{15}C_{11}x^2y^2z^2w - \frac{1}{4}C_{15}C_{10}x^2y^2z^2 \\
& -\frac{1}{4}C_{14}C_{12}x^2y^2zw^2 - \frac{1}{4}C_{14}C_{11}x^2y^2zw - \frac{1}{4}C_{14}C_{10}x^2y^2z - \frac{1}{4}C_{13}C_{12}x^2y^2w^2 \\
& -\frac{1}{4}C_{13}C_{11}x^2y^2w - \frac{1}{4}C_{13}C_{10}x^2y^2 - \frac{1}{2}C_{25}C_{22}x^2yz^2w^2 - \frac{1}{2}C_{25}C_{21}x^2yz^2w \\
& -\frac{1}{2}C_{25}C_{20}x^2yz^2 - \frac{1}{2}C_{24}C_{22}x^2yzw^2 - \frac{1}{2}C_{24}C_{21}x^2yzw - \frac{1}{2}C_{24}C_{20}x^2yz \\
& -\frac{1}{2}C_{23}C_{22}x^2yw^2 - \frac{1}{2}C_{23}C_{21}x^2yw - \frac{1}{2}C_{23}C_{20}x^2y - \frac{1}{2}C_{55}C_{52}x^2z^2w^2 \\
& -\frac{1}{2}C_{55}C_{51}x^2z^2w - \frac{1}{2}C_{55}C_{50}x^2z^2 - \frac{1}{2}C_{54}C_{52}x^2zw^2 - \frac{1}{2}C_{54}C_{51}x^2zw \\
& -\frac{1}{2}C_{54}C_{50}x^2z - \frac{1}{2}C_{53}C_{52}x^2w^2 - \frac{1}{2}C_{53}C_{51}x^2w - \frac{1}{2}C_{53}C_{50}x^2 + \frac{1}{2}C_{35}C_{32}xy^2z^2w^2 \\
& +\frac{1}{2}C_{35}C_{31}xy^2z^2w + \frac{1}{2}C_{35}C_{30}xy^2z^2 + \frac{1}{2}C_{34}C_{32}xy^2zw^2 + \frac{1}{2}C_{34}C_{31}xy^2zw \\
& +\frac{1}{2}C_{34}C_{30}xy^2z + \frac{1}{2}C_{33}C_{32}xy^2w^2 + \frac{1}{2}C_{33}C_{31}xy^2w + \frac{1}{2}C_{33}C_{30}xy^2 \\
& +C_{45}C_{42}xyz^2w^2 + C_{45}C_{41}xyz^2w + C_{45}C_{40}xyz^2 + C_{44}C_{42}xyzw^2 + C_{44}C_{41}xyzw \\
& +C_{44}C_{40}xyz + C_{43}C_{42}xyw^2 + C_{43}C_{41}xyw + C_{43}C_{40}xy + C_{65}C_{62}xz^2w^2
\end{aligned}
$$

$$+C_{65}C_{61}xz^2w+C_{65}C_{60}xz^2+C_{64}C_{62}xzw^2+C_{64}C_{61}xzw+C_{64}C_{60}xz+$$
$$C_{63}C_{62}xw^2+C_{63}C_{61}xw+C_{63}C_{60}x-\frac{1}{2}C_{75}C_{72}y^2z^2w^2-\frac{1}{2}C_{75}C_{71}y^2z^2w$$
$$-\frac{1}{2}C_{75}C_{70}y^2z^2-\frac{1}{2}C_{74}C_{72}y^2zw^2-\frac{1}{2}C_{74}C_{71}y^2zw-\frac{1}{2}C_{74}C_{70}y^2z-\frac{1}{2}C_{73}C_{72}y^2w^2$$
$$-\frac{1}{2}C_{73}C_{71}y^2w-\frac{1}{2}C_{73}C_{70}y^2-C_{85}C_{82}yz^2w^2-C_{85}C_{81}yz^2w-C_{85}C_{80}yz^2$$
$$-C_{84}C_{82}yzw^2-C_{84}C_{81}yzw-C_{84}C_{80}yz-C_{83}C_{82}yw^2-C_{83}C_{81}yw-C_{83}C_{80}y$$
$$-\frac{1}{2}D_{73}D_{72}z^2w^2-D_{83}D_{82}zw^2-\frac{1}{2}D_{73}D_{71}z^2w-D_{83}D_{81}zw-\frac{1}{2}D_{73}D_{70}z^2$$
$$-D_{83}D_{80}z-\frac{1}{2}E_{73}E_{70}w^2-E_{83}E_{80}w\}$$

最後，令係數:

$-\frac{1}{4}C_{15}C_{12}=\alpha_{2222}$、$-\frac{1}{4}C_{15}C_{11}=\alpha_{2221}$、$-\frac{1}{4}C_{15}C_{10}=\alpha_{2220}$、$-\frac{1}{4}C_{14}C_{12}=\alpha_{2212}$、$-\frac{1}{4}C_{14}C_{11}=\alpha_{2211}$、$-\frac{1}{4}C_{14}C_{10}=\alpha_{2210}$、$-\frac{1}{4}C_{13}C_{12}=\alpha_{2202}$、$-\frac{1}{4}C_{13}C_{11}=\alpha_{2201}$、$-\frac{1}{4}C_{13}C_{10}=\alpha_{2200}$、$-\frac{1}{2}C_{25}C_{22}=\alpha_{2122}$、$-\frac{1}{2}C_{25}C_{21}=\alpha_{2121}$、$-\frac{1}{2}C_{25}C_{20}=\alpha_{2120}$、$-\frac{1}{2}C_{24}C_{22}=\alpha_{2112}$、$-\frac{1}{2}C_{24}C_{21}=\alpha_{2111}$、$-\frac{1}{2}C_{24}C_{20}=\alpha_{2110}$、$-\frac{1}{2}C_{23}C_{22}=\alpha_{2102}$、$-\frac{1}{2}C_{23}C_{21}=\alpha_{2101}$、$-\frac{1}{2}C_{23}C_{20}=\alpha_{2100}$、$-\frac{1}{2}C_{55}C_{52}=\alpha_{2022}$、$-\frac{1}{2}C_{55}C_{51}=\alpha_{2021}$、$-\frac{1}{2}C_{55}C_{50}=\alpha_{2020}$、$-\frac{1}{2}C_{54}C_{52}=\alpha_{2012}$、$-\frac{1}{2}C_{54}C_{51}=\alpha_{2011}$、$-\frac{1}{2}C_{54}C_{50}=\alpha_{2010}$、$-\frac{1}{2}C_{53}C_{52}=\alpha_{2002}$、$-\frac{1}{2}C_{53}C_{51}=\alpha_{2001}$、$-\frac{1}{2}C_{53}C_{50}=\alpha_{2000}$、$\frac{1}{2}C_{35}C_{32}=\alpha_{1222}$、$\frac{1}{2}C_{35}C_{31}=\alpha_{1221}$、$\frac{1}{2}C_{35}C_{30}=\alpha_{1220}$、$\frac{1}{2}C_{34}C_{32}=\alpha_{1212}$、$\frac{1}{2}C_{34}C_{31}=\alpha_{1211}$、$\frac{1}{2}C_{34}C_{30}=\alpha_{1210}$、$\frac{1}{2}C_{33}C_{32}=\alpha_{1202}$、$\frac{1}{2}C_{33}C_{31}=\alpha_{1201}$、$\frac{1}{2}C_{33}C_{30}=\alpha_{1200}$、$C_{45}C_{42}=\alpha_{1122}$、$C_{45}C_{41}=\alpha_{1121}$、$C_{45}C_{40}=\alpha_{1120}$、$C_{44}C_{42}=\alpha_{1112}$、$C_{44}C_{41}=\alpha_{1111}$、$C_{44}C_{40}=\alpha_{1100}$、$C_{43}C_{42}=\alpha_{1102}$、$C_{43}C_{41}=\alpha_{1101}$、$C_{43}C_{40}=\alpha_{1100}$、$C_{65}C_{62}=\alpha_{1022}$、$C_{65}C_{61}=\alpha_{1021}$、$C_{65}C_{60}=\alpha_{1020}$、$C_{64}C_{62}=\alpha_{1012}$、$C_{64}C_{61}=\alpha_{1011}$、$C_{64}C_{60}=\alpha_{1010}$、$C_{63}C_{62}=\alpha_{1002}$、$C_{63}C_{61}=\alpha_{1001}$、$C_{63}C_{60}=\alpha_{1000}$、$\frac{1}{2}C_{75}C_{72}=\alpha_{0222}$、$-\frac{1}{2}C_{75}C_{71}=\alpha_{0221}$、$-\frac{1}{2}C_{75}C_{70}=\alpha_{0220}$、$-\frac{1}{2}C_{74}C_{72}=\alpha_{0212}$、$-\frac{1}{2}C_{74}C_{71}=\alpha_{0211}$、$-\frac{1}{2}C_{74}C_{70}=\alpha_{0210}$、$-\frac{1}{2}C_{73}C_{72}=\alpha_{0202}$、$-\frac{1}{2}C_{73}C_{71}=\alpha_{0201}$、$-\frac{1}{2}C_{73}C_{70}=\alpha_{0200}$、$-C_{85}C_{82}=\alpha_{0122}$、$-C_{85}C_{81}=\alpha_{0121}$、$-C_{85}C_{80}=\alpha_{0120}$、$-C_{84}C_{82}=\alpha_{0112}$、$-C_{84}C_{81}=\alpha_{0111}$、$-C_{84}C_{80}=\alpha_{0110}$、$-C_{83}C_{82}=\alpha_{0102}$、$-C_{83}C_{81}=\alpha_{0101}$、$-C_{83}C_{80}=\alpha_{0100}$、$-\frac{1}{2}D_{73}D_{72}=\alpha_{0022}$、$-D_{83}D_{82}=\alpha_{0012}$、$-\frac{1}{2}D_{73}D_{71}=\alpha_{0021}$、$-D_{83}D_{81}=\alpha_{0011}$、$-\frac{1}{2}D_{73}D_{70}=\alpha_{0020}$、$-D_{83}D_{80}=\alpha_{0010}$、$-\frac{1}{2}E_{73}E_{70}=\alpha_{0022}$、$-E_{83}E_{80}=\alpha_{0001}$;

(⇐)因爲

$$\begin{aligned}
\frac{f_{1|234}(x|y,z,w)}{f_{2|134}(y|x,z,w)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-\mu_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_2(x,z,w)}\exp\{-\frac{(y-\mu_2(x,z,w))^2}{2\sigma_2^2(x,z,w)}\}} \\
&= \frac{\sigma_2(x,z,w)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-\mu_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \\
&\quad \exp\{\frac{y^2-2y\mu_2(x,z,w)}{2\sigma_2^2(x,z,w)}\}\exp\{\frac{\mu_2^2(x,z,w)}{2\sigma_2^2(x,z,w)}\}
\end{aligned}$$

$$令 k_1(x,y,z,w) = \frac{-x^2+2x\mu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{y^2-2y\mu_2(x,z,w)}{2\sigma_2^2(x,z,w)}$$

$$\begin{aligned}
= \; & -\frac{x^2}{2}\{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{2122},\alpha_{2121}, \\
& \alpha_{2120},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{2022},\alpha_{2021},\alpha_{2020},\alpha_{2012},\alpha_{2011}, \\
& \alpha_{2010},\alpha_{2002},\alpha_{2001},\alpha_{2000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]\} \\
& +x\{(\alpha_{1222},\alpha_{1221},\alpha_{1220},\alpha_{1212},\alpha_{1211},\alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{1122},\alpha_{1121},\alpha_{1120}, \\
& \alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{1022},\alpha_{1021},\alpha_{1020},\alpha_{1012},\alpha_{1011},\alpha_{1010},\alpha_{1002}, \\
& \alpha_{1001},\alpha_{1000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]*\sigma_1^2(y,z,w)\} \\
& +\frac{y^2}{2}\{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{1122},\alpha_{1221}, \\
& \alpha_{1220},\alpha_{1212},\alpha_{1211},\alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{0222},\alpha_{0221},\alpha_{0220},\alpha_{0212},\alpha_{0211},\alpha_{0210}, \\
& \alpha_{0202},\alpha_{0201},\alpha_{0200})\cdot[(x^2,x,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]\} \\
& -y\{(\alpha_{2122},\alpha_{2121},\alpha_{2120},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{1122},\alpha_{1121},\alpha_{1120}, \\
& \alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{0122},\alpha_{0121},\alpha_{0120},\alpha_{0112},\alpha_{0111},\alpha_{0110},\alpha_{0102}, \\
& \alpha_{0101},\alpha_{0100})\cdot[(x^2,x,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]*\sigma_2^2(x,z,w)\} \\
= \; & h_1(x,z,w)+h_2(y,z,w)
\end{aligned}$$

所以，

$$\frac{f(x|y,z,w)}{f(y|x,z,w)} = V_1(x,z,w)\times[U_1(y,z,w)]^{-1} \tag{46}$$

$$\begin{aligned}
\Rightarrow \; & U_1(y,z,w) = \sigma_1(y,z,w)\exp\{\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0220}y^2z^2+ \\
& \alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0210}y^2z+\alpha_{0202}y^2w^2+\alpha_{0201}y^2w+\alpha_{0200}y^2+ \\
& \alpha_{0122}yz^2w^2+\alpha_{0121}yz^2w+\alpha_{0120}yz^2+\alpha_{0112}yzw^2+\alpha_{0111}yzw+\alpha_{0110}yz+ \\
& \alpha_{0102}yw^2+\alpha_{0101}yw+\alpha_{0100}y\}
\end{aligned} \tag{47}$$

同理，

$$\frac{f(x|y,z,w)}{f(z|x,y,w)} = \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-u_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_3(x,y,w)}\exp\{-\frac{(z-u_3(x,y,w))^2}{2\sigma_3^2(x,y,w)}\}}$$

$$= \frac{\sigma_3(x,y,w)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2xu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-u_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\}$$

$$\exp\{\frac{z^2-2zu_3(x,y,w)}{2\sigma_3^2(x,y,w)}\}\exp\{\frac{u_3^2(x,y,w)}{2\sigma_3^2(x,y,w)}\}$$

令$k_2(x,y,z,w) = \frac{-x^2+2xu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \frac{z^2-2zu_3(x,y,w)}{2\sigma_2^3(x,y,w)}$

$$= -\frac{x^2}{2}\{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{2122},\alpha_{2121},$$
$$\alpha_{2120},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{2022},\alpha_{2021},\alpha_{2020},\alpha_{2012},\alpha_{2011},$$
$$\alpha_{2010},\alpha_{2002},\alpha_{2001},\alpha_{2000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]\}$$
$$+x\{(\alpha_{1222},\alpha_{1221},\alpha_{1220},\alpha_{1212},\alpha_{1211},\alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{1122},\alpha_{1121},\alpha_{1120},$$
$$\alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{1022},\alpha_{1021},\alpha_{1020},\alpha_{1012},\alpha_{1011},\alpha_{1010},\alpha_{1002},$$
$$\alpha_{1001},\alpha_{1000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]*\sigma_1^2(y,z,w)\}$$
$$+\frac{z^2}{2}\{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2122},\alpha_{2121},\alpha_{2120},\alpha_{2022},\alpha_{2021},\alpha_{2020},\alpha_{1222},\alpha_{1221},$$
$$\alpha_{1220},\alpha_{1122},\alpha_{1121},\alpha_{1120},\alpha_{1022},\alpha_{1021},\alpha_{1020},\alpha_{0222},\alpha_{0221},\alpha_{0220},\alpha_{0122},\alpha_{0121},\alpha_{0120},$$
$$\alpha_{0022},\alpha_{0021},\alpha_{0020})\cdot[(x^2,x,1)\otimes(y^2,y,1)\otimes(z^2,z,1)]\}$$
$$-z\{(\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2012},\alpha_{2011},\alpha_{2010},\alpha_{1212},\alpha_{1211},\alpha_{1210},$$
$$\alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1012},\alpha_{1011},\alpha_{1010},\alpha_{0212},\alpha_{0211},\alpha_{0210},\alpha_{0112},\alpha_{0111},\alpha_{0110},\alpha_{0012},$$
$$\alpha_{0011},\alpha_{0010})\cdot[(x^2,x,1)\otimes(y^2,y,1)\otimes(z^2,z,1)]*\sigma_3^2(x,y,w)\}$$
$$= h_3(x,y,w)+h_4(y,z,w)$$

所以，

$$\frac{f(x|y,z,w)}{f(z|x,y,w)} = V_2(x,y,w)\times[U_2(y,z,w)]^{-1} \tag{48}$$

$$\Rightarrow U_2(y,z,w) = \sigma_1(y,z,w)\exp\{\alpha_{0222}y^2z^2w^2+\alpha_{0221}y^2z^2w+\alpha_{0220}y^2z^2+$$
$$\alpha_{0122}yz^2w^2+\alpha_{0121}yz^2w+\alpha_{0120}yz^2+\alpha_{0022}z^2w^2+\alpha_{0021}z^2w+\alpha_{0020}z^2+$$
$$\alpha_{0212}y^2zw^2+\alpha_{0211}y^2zw+\alpha_{0210}y^2z+\alpha_{0112}yzw^2+\alpha_{0111}yzw+\alpha_{0110}yz+$$
$$\alpha_{0012}zw^2+\alpha_{0011}zw+\alpha_{0010}z\} \tag{49}$$

同理，

$$\frac{f(x|y,z,w)}{f(w|x,y,z)} = \frac{\frac{1}{\sqrt{2\pi}\sigma_1(y,z,w)}\exp\{-\frac{(x-u_1(y,z,w))^2}{2\sigma_1^2(y,z,w)}\}}{\frac{1}{\sqrt{2\pi}\sigma_4(x,y,z)}\exp\{-\frac{(w-u_4(x,y,z))^2}{2\sigma_4^2(x,y,z)}\}}$$

$$= \frac{\sigma_4(x,y,z)}{\sigma_1(y,z,w)}\exp\{-\frac{x^2-2xu_1(y,z,w)}{2\sigma_1^2(y,z,w)}\}\exp\{\frac{-u_1^2(y,z,w)}{2\sigma_1^2(y,z,w)}\} \exp\{\frac{w^2-2wu_4(x,y,z)}{2\sigma_4^2(x,y,z)}\}\exp\{\frac{u_4^2(x,y,z)}{2\sigma_4^2(x,y,z)}\}$$

令$k_3(x,y,z,w) = \dfrac{-x^2+2xu_1(y,z,w)}{2\sigma_1^2(y,z,w)} + \dfrac{w^2-2wu_4(x,y,z)}{2\sigma_2^4(x,y,z)}$

$$= -\frac{x^2}{2}\{-2(\alpha_{2222},\alpha_{2221},\alpha_{2220},\alpha_{2212},\alpha_{2211},\alpha_{2210},\alpha_{2202},\alpha_{2201},\alpha_{2200},\alpha_{2122},\alpha_{2121}, \alpha_{2120},\alpha_{2112},\alpha_{2111},\alpha_{2110},\alpha_{2102},\alpha_{2101},\alpha_{2100},\alpha_{2022},\alpha_{2021},\alpha_{2020},\alpha_{2012},\alpha_{2011}, \alpha_{2010},\alpha_{2002},\alpha_{2001},\alpha_{2000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]\}$$

$$+x\{(\alpha_{1222},\alpha_{1221},\alpha_{1220},\alpha_{1212},\alpha_{1211},\alpha_{1210},\alpha_{1202},\alpha_{1201},\alpha_{1200},\alpha_{1122},\alpha_{1121},\alpha_{1120}, \alpha_{1112},\alpha_{1111},\alpha_{1110},\alpha_{1102},\alpha_{1101},\alpha_{1100},\alpha_{1022},\alpha_{1021},\alpha_{1020},\alpha_{1012},\alpha_{1011},\alpha_{1010},\alpha_{1002}, \alpha_{1001},\alpha_{1000})\cdot[(y^2,y,1)\otimes(z^2,z,1)\otimes(w^2,w,1)]*\sigma_1^2(y,z,w)\}$$

$$-\frac{w^2}{2}\{-2(\alpha_{2222},\alpha_{2212},\alpha_{2202},\alpha_{2122},\alpha_{2112},\alpha_{2102},\alpha_{2022},\alpha_{2012},\alpha_{2002},\alpha_{1222},\alpha_{1212}, \alpha_{1202},\alpha_{1122},\alpha_{1112},\alpha_{1102},\alpha_{1022},\alpha_{1012},\alpha_{1002},\alpha_{0222},\alpha_{0212},\alpha_{0202},\alpha_{0122},\alpha_{0112}, \alpha_{0102},\alpha_{0022},\alpha_{0012},\alpha_{0002})\cdot[(x^2,x,1)\otimes(y^2,y,1)\otimes(z^2,z,1)]\}$$

$$+w\{(\alpha_{2221},\alpha_{2211},\alpha_{2201},\alpha_{2121},\alpha_{2111},\alpha_{2101},\alpha_{2021},\alpha_{2011},\alpha_{2001},\alpha_{1221},\alpha_{1211},\alpha_{1201}, \alpha_{1121},\alpha_{1111},\alpha_{1101},\alpha_{1021},\alpha_{1011},\alpha_{1001},\alpha_{0221},\alpha_{0211},\alpha_{0201},\alpha_{0121},\alpha_{0111},\alpha_{0101},\alpha_{0021}, \alpha_{0011},\alpha_{0001})\cdot[(x^2,x,1)\otimes(y^2,y,1)\otimes(z^2,z,1)]*\sigma_3^2(x,y,w)\}$$

$$= h_5(x,y,z)+h_6(y,z,w)$$

所以，

$$\frac{f(x|y,z,w)}{f(w|x,y,z)} = V_3(x,y,z)\times[U_3(y,z,w)]^{-1} \tag{50}$$

$$\Rightarrow U_3(y,z,w) = \sigma_1(y,z,w)\exp\{\alpha_{0222}y^2z^2w^2+\alpha_{0212}y^2zw^2+\alpha_{0202}y^2w^2+ \alpha_{0122}yz^2w^2+\alpha_{0112}yzw^2+\alpha_{0102}yw^2+\alpha_{0022}z^2w^2+\alpha_{0012}zw^2+\alpha_{0002}w^2+ \alpha_{0221}y^2z^2w+\alpha_{0211}y^2zw+\alpha_{0201}y^2w+\alpha_{0121}yz^2w+\alpha_{0111}yzw+\alpha_{0101}yw+ \alpha_{0021}z^2w+\alpha_{0011}zw+\alpha_{0001}w\} \tag{51}$$

接著，根據(47)、(49)式，可得到:

$$
\begin{aligned}
\frac{U_1(y,z,w)}{U_2(y,z,w)} &= \frac{\alpha_{0202}y^2w^2+\alpha_{0201}y^2w+\alpha_{0200}y^2+\alpha_{0102}yw^2+\alpha_{0101}yw+\alpha_{0100}y}{\alpha_{0022}z^2w^2+\alpha_{0021}z^2w+\alpha_{0020}z^2+\alpha_{0012}zw^2+\alpha_{0011}zw+\alpha_{0010}z} \\
&= \frac{f_2(y,w)g_1(y)}{f_1(z,w)g_2(z)} \qquad (52)
\end{aligned}
$$

根據(47)、(51)式，可得到:

$$
\begin{aligned}
\frac{U_1(y,z,w)}{U_3(y,z,w)} &= \frac{\alpha_{0220}y^2z^2+\alpha_{0210}y^2z+\alpha_{0200}y^2+\alpha_{0120}yz^2w+\alpha_{0110}yz+\alpha_{0100}y}{\alpha_{0022}z^2w^2+\alpha_{0012}zw^2+\alpha_{0002}w^2+\alpha_{0021}z^2w+\alpha_{0011}zw+\alpha_{0001}w} \\
&= \frac{f_3(y,z)g_1(y)}{f_1(z,w)g_3(w)} \qquad (53)
\end{aligned}
$$

因此，根據(46)、(48)、(50)、(52)、(53)式可知，皆滿足(i)和(ii)，故得證。

附錄二:定理3.3證明

根據Y.L.Tong(1990)，當隨機變數$\mathbf{X}=(X_1,X_2,...,X_n)'\sim N_n(\tilde{\mu},\mathbf{\Sigma})$，則聯合密度分佈函數型式爲:

$$f(\mathbf{X})=\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}\exp[-\frac{1}{2}(\mathbf{X}-\tilde{\mu})'\mathbf{\Sigma^{-1}}(\mathbf{X}-\tilde{\mu})]$$

其中，

$$\tilde{\mu}=\begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{pmatrix}，\mathbf{\Sigma}=\begin{pmatrix}\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n}\\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2\end{pmatrix}=\begin{pmatrix}\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n}\\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2\end{pmatrix};$$

令

$$\mathbf{\Sigma}^{-1}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{12} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{pmatrix};$$

考慮，$\exp[-\frac{1}{2}(\mathbf{X}-\tilde{\mu})'\mathbf{\Sigma^{-1}}(\mathbf{X}-\tilde{\mu})]$，其中，$(\mathbf{X}-\tilde{\mu})'\mathbf{\Sigma^{-1}}(\mathbf{X}-\tilde{\mu})$是二次式;
所以，$\exp[-\frac{1}{2}(\mathbf{X}-\tilde{\mu})'\mathbf{\Sigma^{-1}}(\mathbf{X}-\tilde{\mu})]$

$$=\exp[-\tfrac{1}{2}\sum_{i=1}^{n}a_{ii}x_i^2+\sum_{i<j}^{n}a_{ij}x_ix_j+\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}x_iu_j-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}u_iu_j];$$

由定理3.2可知，若服從常態分佈，係數比較後可知，除了$\alpha(\sum_{i=1}^{n}x_i^2+x_i+\sum_{i<j}^{n}x_ix_j)$，其餘爲0，所以可將定理3.2結果簡化後可得:

$\forall_{1\leq i\leq n}$，

$\sigma_i^2(x_{-i})=[-2\cdot\alpha_n(x_i^2)]^{-1}$，$\sigma_i^2(x_{-i})>0$

$\mu_i(x_{-i})=[\alpha_n(x_i\cdot(x_{-i},1))\cdot(x_{-i},1)]*\sigma_i^2(x_{-i});$

且係數比較後也可得到:

$\forall_{1\leq i<j\leq n}$，

$a_{ii} = -2\alpha(x_i^2)$、$a_{ij} = \alpha(x_i x_j)$，且

$$\begin{cases} a_{11}\mu_1 + a_{12}\mu_2 + \cdots + a_{1n}\mu_n = \alpha(x_1) \\ a_{12}\mu_1 + a_{22}\mu_2 + \cdots + a_{2n}\mu_n = \alpha(x_2) \\ \vdots \\ a_{1n}\mu_x + a_{2n}\mu_y + \cdots + a_{nn}\mu_n = \alpha(x_n) \end{cases}$$

寫成矩陣型式爲:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \alpha(x_1) \\ \alpha(x_2) \\ \vdots \\ \alpha(x_n) \end{pmatrix};$$

因此，可得:

$$\tilde{\mu} = (\Sigma^{-1})_{ij}\alpha(x_i)\text{，}\Sigma = {(\Sigma^{-1})_{ij}}^{-1};$$

其中: $$(\Sigma^{-1})_{ij} \begin{cases} -\alpha(x_i x_j), & i \neq j \\ -2\alpha(x_i^2), & i = j \end{cases}$$

故得證。